Class 3 - Laplace equation in cylindrical coordinates. Magnetic scalar potential. Magnetic monopole

Class material

Exercise 3.1 - Cylinder with a point charge inside

Given a grounded metal cylinder with h height and R radius. We put a point charge q into the middle of the cylinder. What is the potentil inside the cylinder?

(a) Look for the solution for the potential separately below and above the charge! Write the corresponding boundary conditaions for these potentials!

(b) Give the general solution in cylindrical coordinates!

(c) Determine the expansion coefficients from the boundary conditions!

Exercise 3.2 - Finite cylinder with potential on the shield

Given a closed cylinder along the z axis with height h and radius R. The base of the cylinder is at the $z = 0$ plane. We ground the base and the top of the cylinder, on the walls the potential given by a $V(\phi, z)$ function.

(a) Write the separated form of the Laplace equation and show that it leads to the differential equation of the modified Bessel functions!

(b) Give the separated general solution of the equation!

(c) Determine the $\Phi(\vec{r})$ electrostatic potential funtion from the boundary conditions!

(d) Determine the $\Phi(\vec{r})$ electrostatic potential if

$$V(\phi, z) = V_0 \sin\left(\frac{2\pi}{h} z\right) \cos\left(\frac{\phi}{2}\right)$$

(e) Calculate the $\vec{E}(z)$ electric field along the z axis!

Exercise 3.3 - Screening of the magnetic field

From a sphere with radius R_{out} and permeability μ we cut of a sphere with radius R_{in} with the same centre. Outside the sphere we have a homogenous magnetic induction with B magnitude. Calculate the magnetic induction inside the inner sphere! (Screening of the magnetic field.)

(a) Write the equations and the boundary conditions for the magnetic potential in the three different segments!

(b) Give the general solution in spherical coordinates!

(c) Determine the expansion coefficients from the boundary conditions!

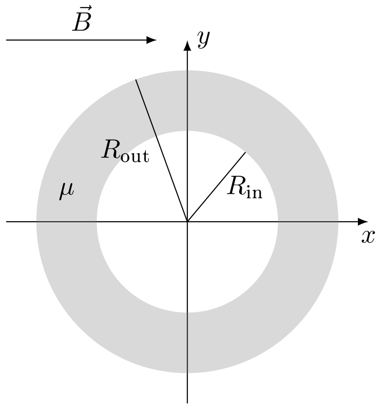

Figure 1

Exercise 3.4 - Semi-infinite solenoid as a magnetic monopole

Consider a semi-infinite solenoid with radius R and srcew density n. In the solenoid we flow I current. Determine the magnetic induction far from the end of the solenoid! Interpret the result as a 'magnetic monopole' and derive the differential equation for it! Show the electric-magnetic symmetry of the Maxwell equations with the magnetic monopole!

(a) Unfold the solenoid to separate rings and give the $B(z)$ magnetic induction of a ring along the axis of the ring for large distances (magnetic dipole)!

(b) Introduce the magnetic scalar potential for the description of the large distance behavior!

(c) Sum up the contributions from the elementary rings to the magnetic potential $\Phi_m(z)$, calculate the potential in the whole space by fitting the complete set:

$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)$$

(d) Calculate the magnetic induction and show that it has a source!

(e) How it modify the Maxwell equations?

(f) Investigate the symmetry between the electric and magnetic sectors of the Maxwell equations!

Homework

The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.

Exercise 3.5 - Surface charge density on azimutal symmetric sphere*

Given a sphere shell with radii a. On this surface the electrostatic potential has cylindrical symmetry: $V(\theta) = V_0 \cos(3\theta)$.

(a) Write the general form of the $\Phi_{\text{in}}(\vec{r})$ potential inside the sphere!

(b) Write the general form of the $\Phi_{\text{out}}(\vec{r})$ potential outside the sphere!

(c) Determine the $\Phi_{\text{in}}(\vec{r})$ and $\Phi_{\text{out}}(\vec{r})$ potentials from the boundary conditions!

(d) Give the surface charge density on the surface of the sphere!

Exercise 3.6 - Coaxial cylinders*

Given two coaxial infinite cylinder along the z axis with radii R_1 and R_2. The potential on the inner one is $V_1(\phi)$ and on the outer one is $V_2(\phi)$.

(a) Give the $\Phi(r, \phi)$ electrostatic potential inside the inner cylinder!

(b) Give the $\Phi(r, \phi)$ electrostatic potential outside the cylinders!

(c) Give the $\Phi(r, \phi)$ electrostatic potential between the cylinders!

(d) Give the surface charge density on the the inner cylinder!

(e) Answer the previous questions if:

$$V_1(\phi) = V_0 \cos\phi$$
$$V_2(\phi) = 2V_0 \cos^2\phi$$

Exercise 3.7 - Square tube around a point charge*

Given a square metal tube along the z axis with side length a. There is a point charge q in the origin of the coordinate system.

(a) Give the $\Phi(\vec{r})$ potential inside the tube!

(b) Give the behaviour of the $\Phi(\vec{r})$ potential far from the point charge!

(c) Give the $\vec{E}(z)$ electric field along the z axis!

(d) Draw the electric field lines describing the field!

Hint:

$$\delta(x)\delta(y) = \left(\frac{2}{a}\right)^2 \sum_{n,m=0}^{\infty} \cos\left(\frac{(2n+1)\pi x}{a}\right) \cos\left(\frac{(2m+1)\pi y}{a}\right)$$

These problems are for further practice and to have some fun!

Exercise 3.8 - Dirac monopole

At the origin there is a magnetic monopole with q_m charge and at $z = a$ there is an electric pointcharge q_e. The magnetic field of the monopole is:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\vec{e}_r \ .$$

(a) Give the $\vec{S}(\vec{r})$ Poynting vector in the whole space!

(b) Give the momentum density of the electromagnetic field!

(c) Give the angular momentum density of the electromagnetic field!

(d) Give the total angular momentum of the electromagnetic field!

(e) Show that, if the angular momentum is quantised (multiples of $\hbar$), then q_e and q_m are also quantised!

Some remarks:
This model of the magnetic monopole was invented by Paul A. M. Dirac(Nobel prize in 1933) in 1931[1], when he was 29 years old. The relevance of this model is that, it gives a hypotetic description of the charge quantisation.

Exercise 3.9 - Multipole (Jackson 6.5)

A localized electric charge distribution produces an electric field $\mathbf{E} = -\nabla\Phi$. Into this field is placed a small localized time-independent current density $\mathbf{J}(\mathbf{x})$, which generates a magnetic field $\mathbf{H}$.

(a) Show that the momentum of these electromagnetic fields,

$$\mathbf{P}_{\text{fied}} = \epsilon_0 \int_V \mathrm{d}^3x \mathbf{E}\times\mathbf{B} = \epsilon_0\mu_0 \int_V \mathrm{d}^3x \mathbf{E}\times\mathbf{H}$$

can be transformed to

$$\mathbf{P}_{\text{fied}} = \frac{1}{c^2}\int_V \mathrm{d}^3x \Phi\mathbf{J}$$

provided the product $\Phi\mathbf{H}$ falls off rapidly enough at large distances. How rapidly is 'rapidly enough'?

(b) Assuming that the current distribution is localized to a region small compared to the scale of variation of the electric field, expand the electrostatic potential in a Taylor series and show that

$$\mathbf{P}_{\text{fied}} = \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m}$$

where $\mathbf{E}(0)$ is the electric field at the current distribution and $\mathbf{m}$ is the magnetic moment,

$$\mathbf{m} = \frac{1}{2}\int \mathrm{d}^3x'\mathbf{x}'\times\mathbf{J}(\mathbf{x}')$$

caused by the current.

(c) Suppose the current distribution is placed instead in a *uniform* electric field $\mathbf{E}_0$ (filling all space). Show that, no matter how complicated is the localized $\mathbf{J}$, the result in part (a) is argumented by a surface integral contribution from infinity equal to minus one-third of the result of part (b), yielding

$$\mathbf{P}_{\text{fied}} = \frac{2}{3c^2}\mathbf{E}_0\times\mathbf{m}$$

Compare this result with that obtained by working directly with the original form of the momentum from part (a) and the considerations at the end *Jackson Sec.5.6.*

[1] Dirac, P. A. M. (1931). Quantised singularities in the electromagnetic field. *Proc. R. Soc. Lond. A,* 133(821), 60-72.

Exercise 3.10 - Rectangular capacitor (Jackson 6.13)

A parallel plate capacitor is formed of two flat rectangular perfectly conducting sheets of dimension a and b separated by a distance d small compared to a or b. Current is fed in and taken out uniformly along the adjacent edges of length b. With the input current and voltage defined at this end of the capacitor, calculate the input impedance or admittance using the field concepts

$$R = \frac{1}{|I_i|^2}\left\{\mathrm{Re}\int_V \mathrm{d}^3x\, \mathbf{J}^* \cdot \mathbf{E} \oint_{S-S_i} \mathrm{d}A\, \mathbf{S}\cdot\mathbf{n} + 4\omega\, \mathrm{Im}\int_V \mathrm{d}^3x\, (w_m - w_e)\right\}$$
$$X = \frac{1}{|I_i|^2}\left\{4\omega\, \mathrm{Re}\int_V \mathrm{d}^3x\, (w_m - w_e) - \mathrm{Im}\int_V \mathrm{d}^3x\, \mathbf{J}^* \cdot \mathbf{E}\right\}$$

where $w_e(w_m)$ is the electric(magnetic) energy density defined as:

$$w_e = \frac{1}{4}\mathbf{E}\mathbf{D}^* \qquad w_m = \frac{1}{4}\mathbf{B}\mathbf{H}^*$$

and $Z = R + \imath X$.

(a) Calculate the electric and magnetic fields in the capacitor correct to second order in powers of the frequency, but neglecting fringing fields.

(b) Show that the expansion of the reactance in powers of the frequency to an appropriate order is the same as that obtained for a lumped circuit consisting of a capacitance $C = \epsilon_0 ab/d$ in series with an inductance $L = \mu_0 ad/3b$

Exercise 3.11 - Circular capacitor (Jackson 6.14)

An ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$ is connected to a current source by axial leads, as shown in the sketch. The current in the wire is $I(t) = I_0 \cos \omega t$.

(a) Calculate the electric and magnetic fields between the plates to second order in powers of the frequency (or wave number), neglecting the effect of fringing field.

(b) Calculate the volume integrals of w_e and w_m that enter the feinition of the reactance X to second order in ω. Show that in terms of the input current I_i, defined by $I_i = -\imath\omega Q$, where Q is the *total charge* on one plate, these energies are

$$\int \mathrm{d}^3x w_e = \frac{1}{4\pi\epsilon_0}\frac{|I_i|^2 d}{\omega^2 a^2}\,, \qquad \int \mathrm{d}^3x w_m = \frac{\mu_0}{4\pi}\frac{|I_i|^2 d}{8}\left(1 + \frac{\omega^2 a^2}{12c^2}\right)$$

(c) Show that the equivalent series circuit has $C \simeq \pi\epsilon_0 a^2/d$, $L \simeq \mu_0 d/8\pi$, and that an estimate for the resonant frequency of the system is $\omega_{textres} = 2\sqrt{2}c/a$. Compare with the first root of the $J_0(x)$ Bessel function.

Exercise 3.12 - Magnetic monopole (Jackson 6.16)

(a) Calculate the force in newtons acting on a Dirac monopole of the minimum magnetic charge located a distance 0.5 Åfrom and in the median plane of a magnetic dipole with dipole moment equal to one nuclear magneton ($e\hbar/2m_p$).

(b) Compare the force in part (a) with atomic forces such as the direct electrostatic force between charges (at the same separation), the spin-orbit force, the hyperfine interaction. Comment on the question of binding of magnetic monopoles to nucleio with magnetic moments. Assum that the monopole mass is at least that of a proton.

Reference: D.Sivers, *Phys. Rev.* **D2**, 2048 (1970).

Exercise 3.13 - Magnetic monopole (Jackson 6.17)

(a) For a particle possessing both electric and magnetic charges, show that the generalization of the Lorentz force in vacuum is

$$\mathbf{F} = q_e\mathbf{E} + q_m\mathbf{B}/\mu_0 + q_e\mathbf{v} \times \mathbf{B} - q_m\mathbf{v} \times \epsilon_0\mathbf{E}$$

(b) Show that this expression for the force is invariant under a duality transformation of both fields $(\mathbf{E}, Z_0\mathbf{H})$, $(Z_0\mathbf{D}, \mathbf{B})$ and charges $(Z_0\rho_e, \rho_m)$, $(Z_0\mathbf{J}_e, \mathbf{J}_m)$, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the vacuum impedance.

(c) Show that the Dirac quantization condition

$$\frac{eg}{4\pi\hbar} = \frac{n}{2} \qquad n \in \mathbb{Z}$$

where e and g are the electric and magnetic charges, is generalized for two particles possessing electric and magnetic charges e_1, g_1 and $e_2, g2$, respectively, to

$$\frac{e_1g_2 - e_2g_1}{4\pi\hbar} = \frac{n}{2} \qquad n \in \mathbb{Z}$$

and that the relation is invariant under a duality transformation of the charges.

Exercise 3.14 - Magnetic monopole (Jackson 6.18)

Consider the Dirac expression

$$\mathbf{A}(\mathbf{x}) = \frac{g}{4\pi}\int_L \frac{\mathrm{d}\mathbf{l}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}')|^3}$$

for the vector potential of a magnetic monopole and its associated string L. Suppose for definiteness that the monopole is located at the origin and the string along the negative z axis.

(a) Calculate $\mathbf{A}$ explicitly and show that in spherical coordinates it has components

$$A_r = 0\ , \qquad A_\theta = 0\ , \qquad A_\phi = \frac{g(1-\cos\theta)}{4\pi r\sin\theta} = \left(\frac{g}{4\pi r}\right)\tan\left(\frac{\theta}{2}\right)$$

(b) Varify that $\mathbf{B} = \nabla \times \mathbf{A}$ is the Coulomb-like field of a point charge, except perhaps at $\theta = \pi$.

(c) With the $\mathbf{B}$ determined in part (b), evaluate the total magnetic flux passing through the circular loop of radius $R\sin\theta$ shown in the figure. Consider $\theta < \pi/2$ and $\theta > \pi/2$ separately, bit always calculate the upward flux.

(d) From $\oint \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ around the loop, determine the total magnetic fluc through the loop. Compare the result with that found in part (c). Show that they are equal for $0 < \theta < \pi/2$, but we have a *constant* difference for $\pi/2 < \theta < \pi$. Interpret this difference.