Mathematical Supplement for Electrodynamics 2

Gábor Takács

26th January 2019

1 Legendre polynomials

Legendre's differential equation

$$\frac{d}{dx}(1-x^2)\frac{dP}{dx}+\nu(\nu+1)P=0 \tag{1}$$

Normalisation of solutions:

$$P_l(1)=1$$

Rodrigues formula:

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

Generating function of Legendre polynomials:

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{l=0}^{\infty}t^l P_l(x)$$

Orthogonality relation:

$$\int_{-1}^{1}dx P_{l'}(x)P_l(x)=\frac{2}{2l+1}\delta_{ll'}$$

2 Associated Legendre functions

Differential equation:

$$\frac{d}{dx}(1-x^2)\frac{dP}{dx}+\left[\nu(\nu+1)-\frac{m^2}{1-x^2}\right]P=0 \tag{2}$$

Associated Legendre functions:

$$P_l^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

for $0\leq m$. Extension to $m<0$:

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

Fundamental properties:

1. $P_l^{m=0}(x)=P_l(x)$
2. $P_l^m(x)=0$, $m>l$.

3. Orthogonality:
$$\int_{-1}^{+1} dx P_l^m(x) P_{l'}^m(x) = 0$$
for $l \neq l'$.

4. Normalisation:
$$\int_{-1}^{+1} dx P_l^m(x) P_l^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}$$

3 Spherical harmonics

Definition:

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

Orthogonality relation:

$$\int d\Omega Y_{lm}(\theta, \phi)^* Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

Addition theorem:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

where γ is the angle between direction parameterised by polar angles θ, ϕ and θ', ϕ'.

4 Bessel functions

4.1 Bessel functions of the first kind

Bessel's differential equations:

$$J''(x) + \frac{1}{x} J'(x) + \left(1 - \frac{\nu^2}{x^2}\right) J(x) = 0$$

Independent solutions for $\nu \notin \mathbb{N}$:

$$\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k} \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k} \end{aligned} \tag{3}$$

and the series are absolute convergent for all $x \in \mathbb{C}$-.

If $\nu = m \in \mathbb{N}$, the two solutions above are related

$$J_{-m}(x) = (-1)^m J_m(x)$$

and the second independent solution is given by the Neumann function:

$$N_\nu(x) = \frac{J_\nu(x) \cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}$$

which has a finite limit for ν integre. J_ν and N_ν form a basis for all ν.

Hankel functions

$$H_\nu^{(1,2)}(x) = J_\nu(x) \pm i N_\nu(x)$$

All functions $\Omega = J, N, H^{(1)}$ and $H^{(2)}$ satisfy

$$\begin{aligned}
\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) &= \frac{2\nu}{x}\Omega_\nu(x) \\
\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) &= 2\frac{d\Omega_\nu(x)}{dx}
\end{aligned}$$

Integral representation:

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^\nu \int_{-1}^{+1} (1-t^2)^{\nu-1/2} e^{ixt} dt \qquad \nu > -1/2$$

For small x

$$\begin{aligned}
J_\nu(x) &\to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \\
N_\nu(x) &\to \begin{cases} \frac{2}{\pi}\left(\log\frac{x}{2} + \gamma\right) & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}
\end{aligned}$$

where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.5772\ldots$$

is the Euler-Mascheroni constant. Their large x asymptotic behaviour is

$$\begin{aligned}
J_\nu(x) &\to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
N_\nu(x) &\to \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)
\end{aligned}$$

4.2 Bessel functions of the second kind

Their differential equation is

$$Y''(x) + \frac{1}{x}Y'(x) - \left(1 + \frac{\nu^2}{x^2}\right)Y(x) = 0$$

which has solutions $I_{\pm\nu}(x)$ where

$$\begin{aligned}
I_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{k=1}^{\infty}\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k} \\
&= i^{-\nu} J_\nu(ix)
\end{aligned}$$

and the complex power is specified as:

$$i^{-\nu} := e^{-i\frac{\pi}{2}\nu}$$

For $\nu = m \in \mathbb{Z}$ one has $I_m \equiv I_{-m}$, and the other independent solution can be written as

$$\begin{aligned}
K_m(x) &= \lim_{\nu\to m} K_\nu(x) \\
K_\nu(x) &= \frac{\pi}{2}\frac{I_\nu(x) - I_{-\nu}(x)}{\sin\nu\pi}
\end{aligned}$$

Relation to Hankel functions:

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

where

$$i^{\nu+1} := e^{i\frac{\pi}{2}(\nu+1)}$$

These functions satisfy

$$\begin{aligned}
\frac{d}{dx}\left(x^\nu I_\nu(x)\right) &= x^\nu I_{\nu-1}(x) \\
\frac{d}{dx}\left(x^{-\nu} I_\nu(x)\right) &= x^{-\nu} I_{\nu+1}(x) \\
\frac{\nu}{x} I_\nu(x) + I_\nu'(x) &= I_{\nu-1}(x) \\
-\frac{\nu}{x} I_\nu(x) + I_\nu'(x) &= I_{\nu+1}(x) \\
I_{\nu-1}(x) - I_{\nu+1}(x) &= \frac{2\nu}{x} I_\nu(x) \\
I_{\nu-1}(x) + I_{\nu+1}(x) &= 2\frac{dI_\nu(x)}{dx}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}\left(x^\nu K_\nu(x)\right) &= -x^\nu K_{\nu-1}(x) \\
\frac{d}{dx}\left(x^{-\nu} K_\nu(x)\right) &= -x^{-\nu} K_{\nu+1}(x) \\
\frac{\nu}{x} K_\nu(x) + K_\nu'(x) &= -K_{\nu-1}(x) \\
-\frac{\nu}{x} K_\nu(x) + K_\nu'(x) &= -K_{\nu+1}(x) \\
K_{\nu-1}(x) - K_{\nu+1}(x) &= -\frac{2\nu}{x} K_\nu(x) \\
K_{\nu-1}(x) + K_{\nu+1}(x) &= -2\frac{dK_\nu(x)}{dx}
\end{aligned}$$

Integral representation:

$$\begin{aligned}
I_\nu(x) &= i^{-\nu} J_\nu(ix) \\
&= \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^\nu \int_{-1}^{+1} (1-t^2)^{\nu-1/2} e^{-xt} dt \qquad x>0\ ,\ \nu>-1/2 \\
K_\nu(x) &= \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^\nu \int_{1}^{\infty} (t^2-1)^{\nu-1/2} e^{-xt} dt \qquad x>0\ ,\ \nu>-1/2
\end{aligned}$$

and asymptotic behaviour:

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + O(1/x)\right)$$

4.3 Roots of the Bessel functions

The equation

$$J_\nu(x) = 0$$

has infinitely many solutions:

$$x_{\nu n} \qquad n = 1, 2, \ldots$$

From the asmptotics of J_ν , the roots far from the origin satisfy:

$$x_{\nu n} \sim n\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}$$

Approximate values for a few cases:

$\nu \backslash n$	1	2	3	4	5	6
0	2.40483	5.52008	8.65373	11.7915	14.9309	18.0711
1	3.83171	7.01559	10.1735	13.3237	16.4706	19.6159
2	5.13562	8.41724	11.6198	14.796	17.9598	21.117
3	6.38016	9.76102	13.0152	16.2235	19.4094	22.5827

4.4 An important integral and orthogonality relation

If for a fixed a ξ satisfies

$$J_\nu(\xi a) = 0$$

then

$$\int_0^a x\,[J_\nu(\xi x)]^2\,dx = \frac{a^2}{2}\,[J_{\nu+1}(\xi a)]^2$$

and the orthogonality relation on the interval $[0, a]$ is

$$\int_0^a d\rho\rho J_\nu(x_{\nu n}\rho/a) J_\nu(x_{\nu n'}\rho/a) = \frac{a^2}{2}\,[J_{\nu+1}(x_{\nu n})]^2\,\delta_{nn'}$$

4.5 Hankel transformation

In the limit of a half-infinite line

$$a \to \infty$$

the orthogonality relation becomes:

$$\int_0^\infty d\rho\rho J_\nu(k\rho) J_\nu(k'\rho) = \frac{1}{k}\delta(k - k')$$

If f is a function satisfying

$$\int_0^\infty d\rho\rho^{1/2}\,|f(\rho)| < \infty$$

then it can be represented as:

$$f(\rho) = \int_0^\infty dk\,kF_\nu(k) J_\nu(k\rho)$$

where $F_\nu(k)$ is the Hankel transform

$$F_\nu(k) = \int_0^\infty d\rho\,\rho f(\rho) J_\nu(k\rho)$$

This is the analogue of the Fourier transform on the half line, and it is well-defined for any fixed $\nu > -1/2$.

5 Some useful relations for Legendre and Bessel functions

1. When computing the electric field near a sharp edge we used

$$P_\nu(\cos\theta) \sim J_0\left((2\nu+1)\sin\frac{\theta}{2}\right)$$

which is true for $\theta < 1$ and large ν, where $P_\nu(x)$ is the solution of Legendre's equation which is regular at $x = 1$
$$P_\nu(1) = 1$$
The other important fact is if we define ν_0 as the smallest ν for which

$$P_\nu(\cos\beta) = 0$$

then it has the asymptotic behaviour

$$\nu_0 \simeq \left[2\ln\left(\frac{2}{\pi-\beta}\right)\right]^{-1}$$

for small $\pi - \beta$.

2. In the derivation of Cherenkov's radiation we used the identity

$$\int_{-\infty}^{\infty} ds \frac{e^{ist}}{\sqrt{s^2+1}} = 2K_0(|t|)$$

6 Electromagnetic field of an arbitrarily moving point charge

$$\begin{aligned}
\vec{E}(t,\vec{x}) &= \frac{q}{4\pi\epsilon_0}\left(1-\vec{\beta}(\bar{t})^2\right)\frac{\vec{R}-R\vec{\beta}(\bar{t})}{\left(R-\vec{R}\cdot\vec{\beta}(\bar{t})\right)^3} + \frac{q\mu_0}{4\pi}\frac{\vec{R}\times\left[\left(\vec{R}-R\vec{\beta}(\bar{t})\right)\times\vec{a}(\bar{t})\right]}{\left(R-\vec{R}\cdot\vec{\beta}(\bar{t})\right)^3} \\
\vec{H}(t,\vec{x}) &= \frac{1}{Z_0}\hat{R}\times\vec{E}(t,\vec{x})
\end{aligned}$$