A1

In class we learned about the two dimensional harmonic oscillator, given by the Hamiltonian:

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) \tag{1}$$

We constructed the 2×2 matrix A_{jk} through

$$A_{jk} = \frac{1}{2}\left(\frac{1}{m}p_i p_j + m\omega^2 x_i x_j\right) \tag{2}$$

and the operators

$$S_1 = \frac{A_{12}}{\omega} \qquad S_2 = \frac{A_{22} - A_{11}}{2\omega} \qquad S_3 = \frac{L}{2} = \frac{1}{2}(xp_y - yp_x) \tag{3}$$

In class we showed that $\{S_1, S_2\} = S_3$.

(a) Now show that

$$\{S_3, S_1\} = S_2 \qquad \{S_2, S_3\} = S_1 \tag{4}$$

You can use either the methods used in class (expressing the Poisson brackets using the Leibniz rule), or perhaps also the direct definition, which for two functions $f(x, y, p_x, p_y)$, $g(x, y, p_x, p_y)$ with 2 degrees of freedom is

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p_x} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial p_x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial p_y} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial p_y} \tag{5}$$

(b) * Show that $H = 4\omega^2(S_1^2 + S_2^2 + S_3^2)$.

A2

A free particle can move along the x-axis. Its Hamiltonian is trivially

$$H = \frac{p^2}{2m} \tag{6}$$

Consider the following quantity, that depends explicitly on the time:

$$F(p, x, t) = x - \frac{tp}{m} \tag{7}$$

- Calculate the Poisson bracket $\{F, H\}$ (warning: it is non-zero!), and show that it is a constant of motion, to be precise:

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t} \tag{8}$$

- In class we learned that for a constant of motion there exists a corresponding symmetry, generated by the conserved quantity. In order to determine the symmetry generated by F, we have to analyze the following equations, where s is the continuous parameter of the transformation:

$$\frac{dx}{ds} = \{x, F\} \qquad \frac{dp}{ds} = \{p, F\} \tag{9}$$

  Calculate the Poisson brackets on the right-hand side.

- Integrate the equations of b.) with respect to s, and determine the $x(s)$ and $p(s)$ expressions. Let the initial conditions be $x(s = 0) = x_0$ and $p(s = 0) = p_0$.

- You can see that $x(s)$ and $p(s)$ give the usual Galilei transformation rules, that is indeed a symmetry of a system consisting of a free particle.

B1

A particle of mass m can move in the $x-y$ plane where a conservative $V(x,y)$ potential is also present.

(a) Write down the Lagrangian of the system and determine the Hamiltonian as a function of p_x, p_y, x and y.

(b) Write down tha Lagrangian of the system using the r and ϕ polar coordinates.

(c) Determine the Hamiltonian of the system as a function of p_r, p_ϕ, r and ϕ. Show that this "new" Hamiltonian (denoted by H') is equal to the "old" Hamiltonian, one only needs to change the variables.

(d) Express the canonical momenta p_x and p_y as functions of p_r, p_ϕ, r and ϕ.

(e) Show that the Poisson brackets between the variables $\{x, y, p_x, p_y\}$ don't change if we calculate them using the polar version of the canonical coordinates.

(f) Show that in the case of a central potential $(V = V(r))$ p_ϕ is a conserved quantity.

B2

The Hamiltonian of a system with one degree of freedom reads as

$$H = \frac{p^2}{2} - \frac{1}{2q^2} \tag{10}$$

(a) Show that the following (explicitly time-dependent) quantity is a constant of motion, i.e. it's value during the Hamiltonian time evolution is constant:

$$D = \frac{pq}{2} - Ht \tag{11}$$

(b) Consider a possible two-dimensional generalization of the problem:

$$H = |\mathbf{p}|^n - a|\mathbf{r}|^{-n} \tag{12}$$

Here $\mathbf{r}$ and $\mathbf{p}$ are two-dimensional vectors. Show that the following quantity is a constant of motion:

$$D = \frac{\mathbf{p}\cdot\mathbf{r}}{n} - Ht \tag{13}$$

B3

You learned about some important algebraic properties of the Poisson bracktes,

(a) $\{F, G\} =$ - $\{G, F\}$

(b) $\{F, aG + bD\} = a\{F, G\} + b\{F, D\}$, where a and b are real numbers.

(c) $\{F, GD\} = G\{F, D\} + \{F, G\}D$

(d) Jacobi identity $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$

The proof of the first three relations is trivial, but the fourth is quite complicated. By making use of the simplectic matrix J prove the Jacobi identity. Use the antisymmetric property of J.