A1

A charged particle can move in the $x-y$ plane, while a magnetic field that points in direction z is also present. A possible choice for the Hamiltonian of the system is

$$H(x,y,p_x,p_y) = \frac{p_x^2}{2m} + \frac{(p_y - eBx)^2}{2m} \tag{1}$$

(a) Write down the full Hamilton-Jacobi equation for the system.

(b) Following the usual separation , write down the abbreviated Hamilton-Jacobi equation for the S_0 function.

(c) Separate the equation further. Look for the solution in the form

$$S_0(x,y) = S_0(x,\alpha_x) + S_0(y,\alpha_y) \tag{2}$$

Substitute this form into the shortened Hamilton-Jacobi equation.

(d) You can see, that the dependence on y appears only through $\frac{\partial S_y}{\partial y}$, the other terms are independent of y. Therefore we can choose $\frac{\partial S_y}{\partial y} = \alpha_y$ to be a constant. Use this choice. What equation you get for S_x?

(e) Solve the equations, and express the functions S_x and S_y. Write down the full solution $S(x,y,t)$.

A2

A rubber ball of mass m can move along the axis x in a box of length L. At the endpoints of the box the ball bounces back elastically and immediately. Between the two walls the motion of the ball is described by the Hamiltonian

$$H(x,p) = \frac{p^2}{2m} \tag{3}$$

(a) Draw the trajectory of the ball in the phase-plane if the ball has energy E.

(b) Determine the phase-surface bounded by the trajectory. Denote it by $2\pi I$!

(c) Using the dependence of I on E determine the period of the motion.

B1

A particle can move along the x axis, and its Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{\alpha}{|x|} \tag{4}$$

(Remark: This is the special case of the Kepler problem, when the angular momentum of the particle is zero.)

(a) Draw on the phase-plane the bounded trajectories that correspond to negative energies, $E < 0$.

(b) Determine the maximal distance x_{max} as a function of E.

(c) Write down the integral that determines the action variable I.

(d) We know, that

$$\xi = \int_0^1 du \sqrt{1 + \frac{1}{u}} = 2.296... \tag{5}$$

(the integral can be determined analytically, but you don't need to do it now) Using this, determine the function $I(E)$.

(e) Determine the period of the motion as a function of E.

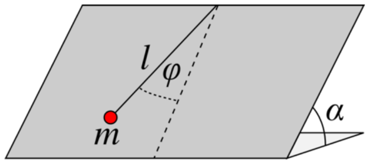
B2

A pendulum is put on a ramp whose slope α can be modified. The length of the pendulum is l, the mass of the body is m. We describe the position of the pendulum by the angle φ. The friction between the body and the ramp is negligible.

(a) Construct the Hamiltonian of the system.

(b) For a given value of α determine the action variable $I(E, \alpha)$ as a function of energy.

(c) The angle-amplitude of the motion was initially A_0, when the slope of the ramp was φ_0. Then the slope of the ramp was changed slowly to φ_1. Using the theorem of adiabatic invariance (problem 5. of class) determine the final angle-amplitude A_1 of the system.



B3

Consider the problem of central motion. The Hamiltonian of the system in polar coordinates is

$$H(r, \varphi, p_r, p_\varphi) = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r) \tag{6}$$

(a) Write down the full Hamilton-Jacobi equation of the system.

(b) Separate the time, and write down the abbreviated Hamilton-Jacobi equation for S_0.

(c) Separate the angle, i.e. search the solution in the form

$$S_0 = S_r(r, L, E) - L\varphi, \tag{7}$$

where L is a constant. Write down the (so called) radial Hamilton-Jacobi equation for S_r.

(d) Determine the integral that expresses S_r.

The integral cannot be evaluated in general. However, many questions can be answered without evaluating it.

(e) At the moment $t = 0$ the particle's distance from the origin is $r = R$ and is at the angle $\varphi = 0$, while it has momenta $p_r = 0$ and $p = L_0$. Using these initial conditions determine the constants L and E.

(f) The canonical coordinates corresponding to E and L are denoted by β_E and β_L. We have not evaluated the function S in a closed fotm, but these constants can be determined in a form, where only an integral over r remains:

$$\begin{aligned} \beta_E &= -t + \int dr \ldots \\ \beta_L &= \varphi + \int dr \ldots \end{aligned} \tag{8}$$

Determine the form of the two integrals!

(g) EXTRA! Let $V(r) = -k/r$. Then the integral for β_L can be evaluated. What do you get?