2. MECHANICS

The purpose of the practice is to experimentally examine the periodic motions learned during our previous studies and often experienced in everyday life, and to examine the practical applicability of the models presented in the theory (ideal spring; pendulum).

THEORY

1. The ideal spring

According to Hooke's Law the restoring force of an ideal spring is proportional to its elongation x:

$F = k\,x$

where $x = \ell - \ell_0$, ℓ is the actual length of the spring, ℓ_0 is its length in relaxed state;

k is the spring constant (or stiffness; dimension: $\frac{N}{m} = \frac{kg}{s^2}$).

For an ideal spring there is no force when $\ell = \ell_0$, i.e. $x = 0$.

When a mass m is suspended at the end of the spring in vertical position, in equilibrium

$k\,x = m\,g.$

The spring constant can be determined by measuring the length of the spring with different loads.

2. Oscillations of a spring

Undamped oscillations

When a mass m is suspended at the end of the spring in horizontal position (neglecting friction) then the equation of motion is

$$m\frac{d^2x}{dt^2} = -k\,x\,.$$

The restoring force kx is negative because the force points always towards the relaxed state $x = 0$, so the direction of the restoring force is always opposite to that of the elongation.

By displacing the mass from the relaxed state $x = 0$ to an initial position $x = x_0$ and/or giving an initial speed v_0 to the mass, it will exhibit simple harmonic motion:

$x(t) = A\cos(\omega t + \varphi_0)$

where A is the amplitude (the maximal elongation) [m];

ω is the angular frequency [s^{-1}]; $\omega = 2\pi f$; $f=1/T$ is the frequency [Hz], T is the time period [s];

$\omega t + \varphi_0$ is the phase; φ_0 is the phase constant [rad].

The speed of the mass is the derivative of $x(t)$:

$v(t) = dx/dt = -A\,\omega\sin(\omega t + \varphi_0)$; the maximal speed (when x = 0): $v_{max} = A\omega$.

The acceleration is the derivative of the speed $v(t)$:

$a(t) = dv/dt = d^2x/dt^2 = -A\,\omega^2\cos(\omega t + \varphi_0).$

Observe that $a(t) = -\omega^2\,x(t).$

Comparing $d^2x/dt^2 = -(k/m)\,x$ from the equation of motion

and $d^2x/dt^2 = -\omega^2\,x$ from the acceleration,

we can see that $\omega^2 = k/m.$

The angular frequency ω is determined by the mass m and the spring constant k:

$\omega = \sqrt{k/m}\,,$

As $\omega = 2\pi/T$, the time period T is

$$T = 2\pi\sqrt{\frac{m}{k}}\,. \qquad [3]$$

The amplitude A and the initial phase φ_0 are determined by the initial conditions x_0 and v_0:

$$\varphi_0 = \text{arc tg}\left(-\frac{v_0}{\omega\,x_0}\right), \quad A = \sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2}\quad .$$

Damped oscillations

If there is also a damping force acting on the body that is <u>proportional to the speed</u>, we have the following equation of motion:

$$m\ d^2x/dt^2 = -k\,x - c\,dx/dt.$$

For relatively small damping factor c the solution is similar to the simple harmonic motion but with exponentially decreasing amplitude:

$$x(t) = A_0\,e^{-\beta t}\cos(\omega\,t + \varphi_0).$$

The angular frequency ω is smaller than that of the undamped oscillation (ω_0):

$$\omega = \sqrt{\omega_0^2 - \beta^2},\quad \text{where } (\omega_0)^2 = k/m \text{ and } 2\beta = c/m.$$

In the case of higher damping factor (i.e. when $\beta \geq \omega_0$) the motion gets aperiodic.

The spring in vertical position, undamped oscillations

Here we must also consider the gravitational force:

$$m\,d^2y/dt^2 = -k\,(y - \ell_0) + mg\,.$$

ℓ_0 is the length of the spring at rest and y is the position of the body measured from the end of the spring. At first sight this equation of motion is different from the equation for the horizontal case. However, by introducing a new variable it can be seen that its solution is also a simple harmonic oscillation but the equilibrium point is shifted.

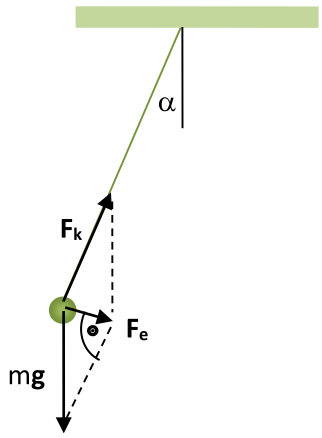
The stable equilibrium position y_E of the body is calculated from the condition that $d^2y/dt^2 = 0$ and it is

$$y_E = \ell_0 + mg/k.$$

The time period of the spring in vertical position equals to that of in horizontal position, only the equilibrium position changes from ℓ_0 to $\ell_0 + mg/k$.

3. The simple pendulum

The equation of motion of a small object m suspended from the end of a light, inextensible string of length L swinging in a <u>vertical</u> plane is the following:



$$m\,a_t = m\,L\frac{d^2\alpha}{dt^2} = -mg\sin\alpha$$

where α is the time dependent angle of the string measured from the equilibrium (vertical) position. As it is a nonlinear differential equation we introduce the following approximation:

$$\sin\alpha \approx \alpha\,,$$

and so the problem is similar to that of the spring:

$$mL\frac{d^2\alpha}{dt^2} = -mg\alpha \quad\rightarrow\quad \frac{d^2\alpha}{dt^2} = -(g/L)\,\alpha$$

$$\alpha(t) = \alpha_{max}\cos(\omega t + \varphi_0) \quad\rightarrow\quad \frac{d^2\alpha}{dt^2} = -\omega^2\alpha$$

$$\rightarrow\qquad \omega = \sqrt{\frac{g}{L}} = \frac{2\pi}{T}.$$

This means that the motion of the pendulum is harmonic with a time period of

$$T = 2\pi\sqrt{\frac{L}{g}}. \qquad [4]$$

The deviation caused by the approximation is 0.05% for α=5°, 1% for α=22° and 18% for α=90°.

MEASUREMENTS

1. Determining the spring constant *k* based on elongation

Tools: – a spring in vertical position,
– a mm scale along it,
– nuts as known masses,
– a PVC rod as a holder of the nuts,
– an object with unknown mass.

Measure m_{PVC}, the mass of the PVC rod (you may also check m_{nut}, the mass of the nuts).
1.1. Read the lowest position of the spring (denoted by *z* in the table)
- without any load;
- with the PVC rod;
- with 1 nut; with 2 nuts; etc;

1.2. Read the lowest position of the spring with the object with unknown mass (add also some nuts if needed).

EVALUATION
The mass of the load m can be expressed in terms of the number of the nuts n:

$$m = m_{PVC} + n\, m_{nut}\ .$$

The elongation x can be expressed in terms of the measured position z:

$$x = z - z_0\ ,\quad \text{where } z_0 \text{ is the position without any load,}$$

so $k\,x = m\,g$: $k\,(z - z_0) = (m_{PVC} + n\, m_{nut})\,g$.

We expect that by plotting z *vs.* n the depicted points fit to a straight line, but we will see that in this case it is not true. In our case $x = 0 \rightarrow F = 0$ is not valid. The reason for this is that there is a force F_0 acting even in the relaxed spring. This has to be included in the equation:

$$k\,x + F_0 = m\,g$$

$$k\,(z - z_0) + F_0 = (m_{PVC} + n\, m_{nut})\,g$$

The measured position z can be expressed as a function of the number of the nuts n:

$$z = \frac{m_{nut}\,g}{k}\,n + \left(\frac{m_{PVC}\,g - F_0}{k} + z_0\right)\ . \qquad [1]$$

Neglecting the points that do not fit to the straight line because of force F_0 the spring constant k can be calculated from the slope

$$a = \frac{m_{nut}\ g}{k}\ . \qquad [2]$$

1.1. Plot the position of the lowest point of the spring z against the number of the nuts n (on an A4 size mm paper, or in Excel).
Based on the graph find the values that fit to a straight line.
Evaluate the measurement using the method of least squares:
choose the values that fit to the line;
calculate the averages $\overline{n},\ \overline{z},\ \overline{n^2},\ \overline{n\cdot z}$;
calculate the slope a;
calculate the value of the spring constant k using equation [2].
1.2. Determine the unknown mass using the diagram.
1.3. OPTIONAL TASK:
Estimate the error of the unknown mass supposing that the reading error is 1 mm.

2. Oscillations of a spring

Tools: the same as for the previous measurement.

2.1. Determining the spring constant *k* based on oscillations

Put three different loads (e.g. 4, 7, 10 nuts) on the PVC rod, suspend it to the spring, pull it, let it oscillate and measure the time of 10 periods.

OPTIONAL TASK: Measure the time of 10 periods with the unknown mass as a load, so that you add also some nuts to get stable oscillations.

2.2. Examining how the time period depends on the amplitude

Measure the time of 10 periods 3 times with the same load so that the amplitude of the oscillations increases. (The amplitude is the difference between the most elongated and the equilibrium positions.)

2.3. OPTIONAL TASK: Damped oscillations

Measure how long it takes for the initial amplitude to be halved. Repeat the measurement with another load.

EVALUATION

2.1. Calculate the spring constant for all cases, using formula [3] for undamped oscillations. Compare the 1+3 values obtained for the spring constant.

OPTIONAL TASK: Determine the unknown mass using the value of the spring constant calculated above, and compare it to the value calculated in **1.2.**

2.2. Explain your observation.

2.3. OPTIONAL TASK: Explain your observation. How does the rate of damping depend on the mass?

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3. The simple pendulum

Tools: – a nut attached to the end of a string (fishing line) as a pendulum,
– stop watch,
– tape measure.

Measure the length of the string L.

3.1. Set the pendulum bob in motion with a small initial angle. Measure the time of 10 periods. Repeat the measurement 5 times.

3.2. Measure the time of 10 periods when the initial angle is increased.

Mandatory task: Make 2 more measurements, it is not necessary to measure the maximal angle.

OPTIONAL TASK: Make 5 measurements, measure the maximal angle.

EVALUATION

3.1. Calculate the average time period $\bar{T}$ and the confidence interval ΔT for P = 95%.

Calculate the mean value of the acceleration of gravity $\bar{g}$ (by substituting the value $\bar{T}$ in [4]).

Calculate the confidence interval Δg using the formula of error propagation.

ΔL is not measured, use 3–5 mm as an estimation.

3.2. Mandatory: Explain your observation.

OPTIONAL TASK: Plot the time period as a function of the maximal angle.

OPTIONAL:

4. Torsion pendulum

A torsion pendulum is analogous to a mass-spring oscillator. Instead of a mass at the end of a helical spring, which oscillates back and forth along a straight line, however, it has a mass at the end of a torsion wire, which rotates back and forth. To set the mass-spring in motion, you displace the mass from its equilibrium position by moving it in a straight line and then releasing it. The helical spring (or gravity, depending on whether or not the system is oriented vertically, and in which direction you displace the mass) exerts a (linear) force to restore the mass to its equilibrium position. To set the torsion pendulum oscillating, you turn the mass (rotate it about its center), and then release it. To do this, you must exert a torque about the bottom of the torsion wire. The torsion wire, in turn, exerts a restoring torque to bring the mass back to its original position.
The time period in this case is

$$T = 2\pi\sqrt{\frac{\Theta}{D}} \qquad [5]$$

D is the direction coefficient (the analogue of k).
Θ is the moment of inertia. In case of a disc

$$\Theta_{disc} = ½\, MR^2.$$

Tools:
– a spring in vertical position,
– plastic box,
– discs of different radii,
– stop watch.

Fasten the box to the spring, rotate it, and measure T_{box}, the time of 1 period.
Attach a disc to the box, and measure $T_{box+disc}$, the time of 1 period.
Measure the radius and the mass of the disc.
OPTIONAL TASK: Make three more measurements attaching another disc, then two discs, etc.

EVALUATION
Calculate the moment of inertia of the box by solving the equation system

$$T_{box} = 2\pi\sqrt{\frac{\Theta_{box}}{D}}$$

$$T_{box+disc} = 2\pi\sqrt{\frac{\Theta_{box} + \Theta_{disc}}{D}}$$

Alternatively, the moment of inertia of the box can be calculated by curve fitting.

REPORT

Introduction

date of the measurement	mane(s) of the student(s)		group No.
2. MECHANICS	name of the teacher		

1. Determining the spring constant *k* based on elongation

Make a sketch of the measurement arrangement and name each object.
On the sketch, draw the forces acting on the “PVC rod + load” system so that the length of the vectors is proportional to the magnitude of the forces.
Describe how to measure, what to change, and what quantity to read.

2. Oscillations of a spring

2.1. Determining the spring constant *k* based on oscillations

2.2. Examining how the time period depends on the amplitude

Make a sketch of the measurement arrangement and name each object.
On the sketch, draw the forces acting on the “PVC rod + load” system at the lowest and at the highest points of the motion so that the length of the vectors is proportional to the magnitude of the forces.
Describe how to measure, what to change, and what quantity to read.

3. The simple pendulum

Make a sketch of the measurement arrangement and name each object.
Describe how to measure, what to change, and what quantity to read.

ENTRANCE TEST

Describe the measurements: tools, instructions, quantities measured and calculated, formulas.

Definitions, formulas:

- Hooke's Law
- harmonic motion
- time period [3], [4]

Calculations:

1. A mass of m is suspended at the end of a spring. ℓ_0 = 22 cm; k = 4.2 N/m. We displace the mass from its equilibrium position by $\Delta\ell$ = 12 cm and let it oscillate. The time of 8 periods is t_8 = 6.4 s.
a) Determine the unknown mass m.
b) What would be the time period by doubling the weight?
c) How would the time period change if we took it to the Moon?

Solution:
ℓ_0 and $\Delta\ell$ are unnecessary data.
Time period: $T = t_8/8 = 0.8$ s.

[3]: $T = 2\pi\sqrt{\frac{m}{k}}$

a) $m = k\,(T/(2\pi))^2 = 4.2 \cdot (0.8/(2\cdot\pi))^2 = 6.8\cdot10^{-2}$ kg = 68 g.

b) $T_{2m} = 2\pi\sqrt{\frac{2m}{k}} = \sqrt{2}\,T = \sqrt{2}\cdot 0.8 = 1.13$ s.

c) The time period does not depend on g so it would not change.

2. Neil Armstrong on his trip to the Moon measured the time period of a simple pendulum made of a 26.0 cm long string with a 12.5 g bob. He got T = 2.50 s.
a) Calculate the gravitational acceleration on the Moon.
b) What time period would he have measured by doubling the weight?
c) What time period would he have measured on the Earth with the same pendulum?

Solution:

[4]: $T = 2\pi\sqrt{\frac{L}{g}}$

a) $g_{Moon} = L\,(2\pi/T)^2 = 0.260 \cdot (2\cdot\pi/2.50)^2 = 1.64$ m/s^2.

b) The time period does not depend on the mass so it would not change.

c) $g_{Earth} = 9.81$ m/s^2, $T_{Earth} = 2\pi\sqrt{\frac{0.260}{9.81}} = 1.02$ s.