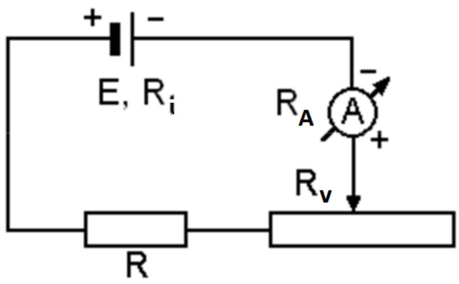
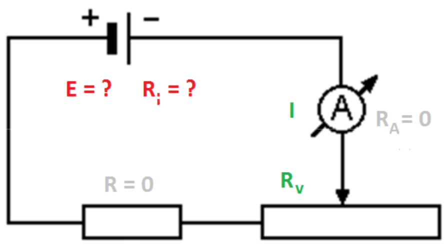
CURVE FITTING

In a series circuit we adjust the resistance and measure the current.
Real measurement: source (E, R_i), constant resistor (R), variable resistor (R_v), ammeter (R_A). Here we neglect R and R_A.





Data

R_v (Ω)	I (A)
100	0.082
200	0.044
250	0.036
300	0.030
400	0.022
600	0.016

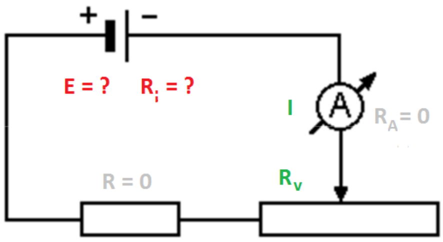
We want to know the value of E and R_i .

Excel trendline:

$$y = 5.8536\ x^{-0.925}$$

$I = 5.8536\ R_v^{\ -0.925}$ $\rightarrow$ E = ?!? R_i = ?!?

it is only useful for iteration in this interval

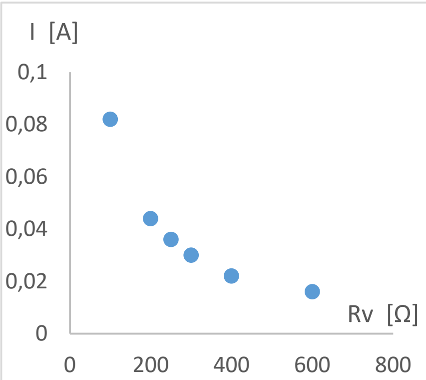


$E = I\,(R_i + R_v)$ →

$$I = \frac{E}{R_i + R_v}$$

x	R_v
y	I
A	E
B	R_i

$$y = \frac{A}{B + x}$$



From theory we know that we have a hyperbola.

$$I = \frac{E}{R_i + R_v}$$

R_v (Ω)	I (A)
100	0.082
200	0.044
250	0.036
300	0.030
400	0.022
600	0.016

$$0.082 = \frac{E}{R_i + 100}$$

$$0.044 = \frac{E}{R_i + 200}$$

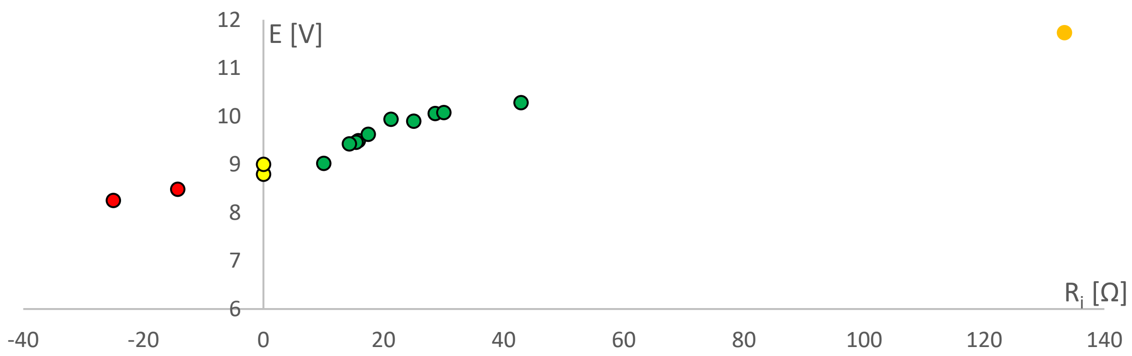
$$0.036 = \frac{E}{R_i + 250}$$

$$0.030 = \frac{E}{R_i + 300}$$

$$0.022 = \frac{E}{R_i + 400}$$

$$0.016 = \frac{E}{R_i + 600}$$

6 equations, 2 unknowns: 15 equation systems!



arithmetic means: $E = 9.57082$ V $R_i = 20.9693\ \Omega$ is this good ?!?

$E = 9.57082 \approx 9.57\ V$; $R_i = 20.9693 \approx 21\ \Omega$

$$I = \frac{E}{R_i + R_V} : \qquad y = \frac{9.57}{21 + x}$$

SciDAVis

mark data,   Plot → Scatter

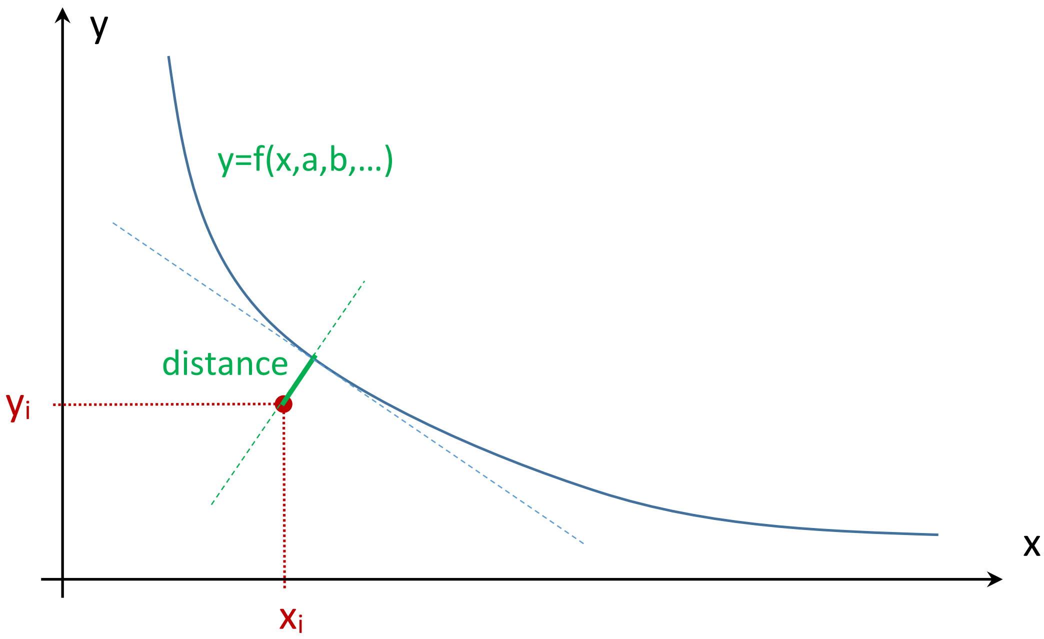
Graph → Add Function   Ctrl + Alt + F

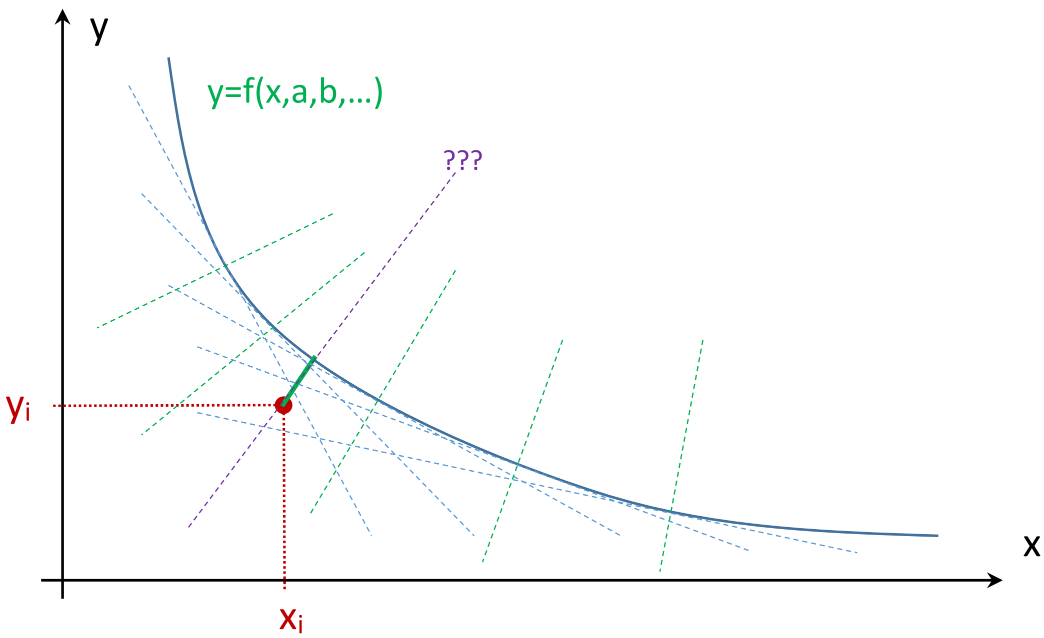
f(x) = 9.57 / (x + 21)

From x = 0 → 90
To x = 1 → 600
Points 100
Ok

better fit?
E = ??? R_i = ???

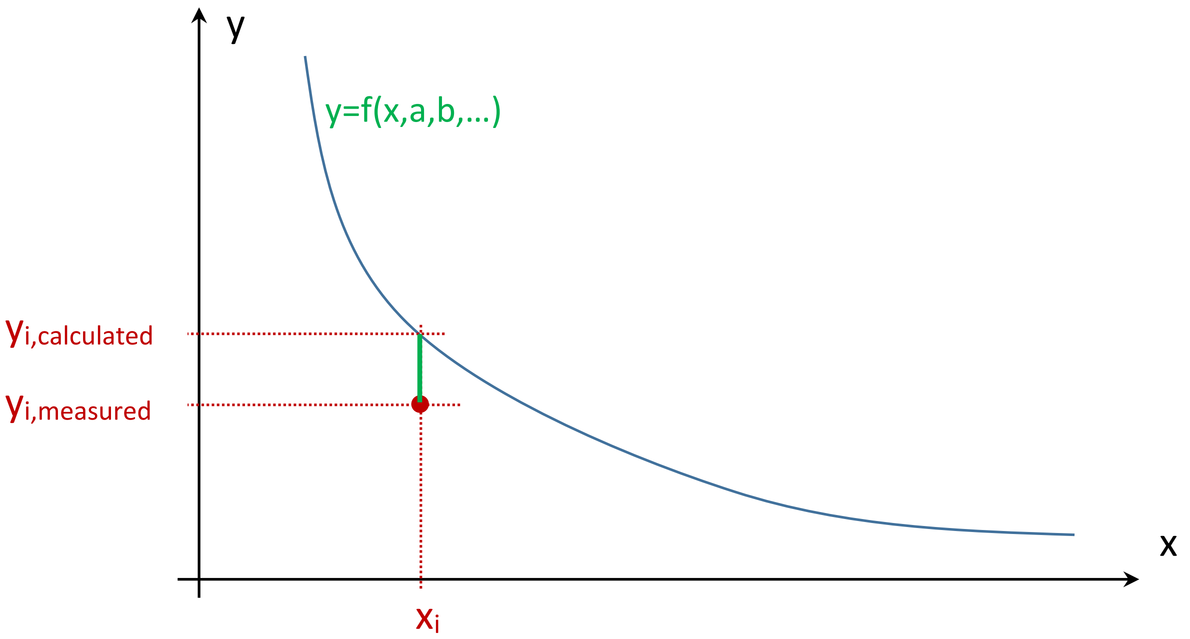
We want minimum distance:





It is not possible to express the distance as a function of the unknown parameters.

The distance will be redefined this way:



$$S = \Sigma\,(\,y_{i,calculated} - y_{i,measured}\,)^2 \;\rightarrow\; \min.$$

$$S = \Sigma\,(\,f(x_i,a,b,\ldots) - y_{i,measured}\,)^2 \;\rightarrow\; \min.$$

“least squares method”

$$y = \frac{A}{x + B} :$$

$$S = \sum \left(\frac{A}{x_i + B} - y_i \right)^2 \rightarrow \text{min.}$$

$$\frac{\partial S}{\partial A} = 2 \sum \left(\frac{A}{x_i + B} - y_i \right) \cdot \left(\frac{1}{x_i + B} \right) = 0$$

$$\frac{\partial S}{\partial B} = 2 \sum \left(\frac{A}{x_i + B} - y_i \right) \cdot \left(- \frac{A}{(x_i + B)^2} \right) = 0$$

i	x_i	y_i
1	100	0.082
2	200	0.044
3	250	0.036
4	300	0.030
5	400	0.022
6	600	0.016

$\rightarrow$ numerical solution gives

$E = 9.480524812$ V $\quad R_i = 15.59110420\ \Omega$

$E \approx 9.48$ V $\quad R_i \approx 15.6\ \Omega$

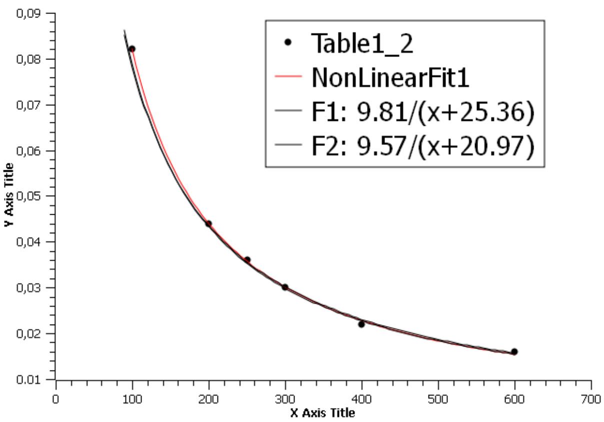
$$I = \frac{E}{R_i + R_V} : \qquad y = \frac{9.48}{15.6 + x}$$

SciDAVis

SciDAVis:

Analysis → Fit wizard   Ctrl + Y

a / (x + b)
Fit >>
Fit



a: E = (9.4805247987374 +/- 0.131725431900025) V
b: R_i = (15.5911037006196 +/- 1.97162054657795) Ω
Chi^2/doF = 2.78453858417807e-07
R^2 = 0.999599250863395

E = (9.48 ± 0.132) V R_i = (15.6 ± 1.97) Ω

relative error:

$$\frac{\Delta E}{E} = \frac{0.132}{9.48} = 0.01389 = 1.389\% \qquad \frac{\Delta R_i}{R_i} = \frac{1.971}{15.59} = 0.1265 = 12.65\% \ !$$

LINEAR: $y = ax + b$

$$S = \sum(ax_i + b - y_i)^2 \quad \rightarrow \quad \text{min.}$$

$$\frac{\partial S}{\partial a} = 2\sum(ax_i + b - y_i) \cdot x_i = 0 \quad (1)$$

$$\frac{\partial S}{\partial b} = 2\sum(ax_i + b - y_i) \cdot 1 = 0 \quad (2)$$

(1): $\sum(ax_i + b - y_i) \cdot x_i = \sum(ax_i^2 + bx_i - x_iy_i) =$

$$= \sum(ax_i^2) + \sum(bx_i) + \sum(-x_iy_i) =$$

$$= a\sum(x_i^2) + b\sum(x_i) - \sum(x_iy_i) = 0$$

(2): $\sum(ax_i + b - y_i) = \sum(ax_i) + \sum(b) + \sum(-y_i) =$

$$= a\sum(x_i) + n \cdot b - \sum(y_i) = 0$$

$$\rightarrow \quad b = \frac{\sum(y_i) - a\sum(x_i)}{n}$$

(1): $a\sum(x_i^2) + \frac{\sum(y_i) - a\sum(x_i)}{n} \cdot \sum(x_i) - \sum(x_iy_i) = 0$

$$a\sum(x_i^2) + \frac{\sum(y_i)\cdot\sum(x_i)}{n} - \frac{a\sum(x_i)\cdot\sum(x_i)}{n} - \sum(x_iy_i) = 0$$

$$a\left(\sum(x_i^2) - \frac{\sum(x_i)\cdot\sum(x_i)}{n}\right) = \sum(x_iy_i) - \frac{\sum(y_i)\cdot\sum(x_i)}{n}$$

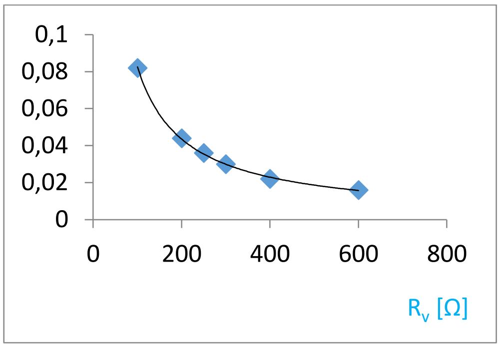
$$a = \frac{\sum(x_iy_i) - \frac{\sum(y_i)\cdot\sum(x_i)}{n}}{\sum(x_i^2) - \frac{\sum(x_i)\cdot\sum(x_i)}{n}} \qquad a = \frac{\frac{\sum(x_iy_i)}{n} - \frac{\sum(x_i)}{n}\frac{\sum(y_i)}{n}}{\frac{\sum(x_i^2)}{n} - \frac{\sum(x_i)}{n}\frac{\sum(x_i)}{n}}$$

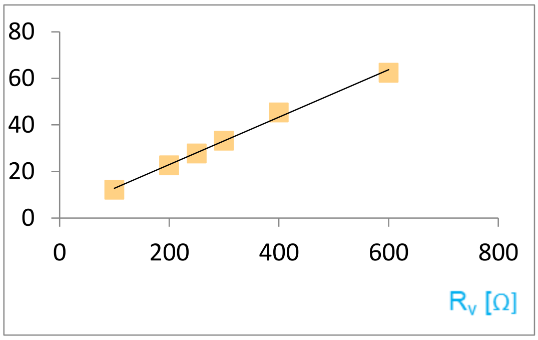
$$a = \frac{\overline{x \cdot y} - \overline{x} \cdot \overline{y}}{\overline{x^2} - \overline{x}^2}$$

$$b = \overline{y} - a \cdot \overline{x}$$

LINEARIZATION

$$I = \frac{E}{R_v + R_i} \quad \longrightarrow \quad f(I) = a\cdot R_v + b \quad ????$$





$$\frac{1}{I} = \frac{R_v + R_i}{E}$$

$$\frac{1}{I} = \frac{1}{E}\cdot R_v + \frac{R_i}{E}$$

$$y' = a \cdot x' + b$$

$$x' = x = R_v$$

$$y' = \frac{1}{I}$$

$$a = \frac{1}{E}$$

$$b = \frac{R_i}{E}$$

$$a = \frac{\overline{x\cdot y} - \bar{x}\cdot\bar{y}}{\overline{x^2} - \bar{x}^2} : \quad \frac{\overline{\left(R_v\cdot\frac{1}{I}\right)} - \overline{R_v}\cdot\overline{\left(\frac{1}{I}\right)}}{\overline{\left(R_v^{\,2}\right)} - \overline{R_v}^{\,2}} = \frac{1}{E} \qquad a = 0.101884 \;\rightarrow\; E = 1/a = 9.815049\ V$$

$$b = \bar{y} - a\cdot\bar{x} : \quad \overline{\left(\frac{1}{I}\right)} - a\cdot\overline{R_v} = \frac{R_i}{E} \qquad b = 2.583663 \;\rightarrow\; R_i = b/a = 25.35878\ \Omega$$

Units!

$\bar{x} = \overline{R_v}$ [Ω]; $\bar{y} = \overline{\left(\frac{1}{I}\right)}$ [1/A]; $\overline{x^2} = \overline{\left(R_v^{\,2}\right)}$ [Ω²]; $\overline{x\cdot y} = \overline{\left(R_v\cdot\frac{1}{I}\right)}$ [Ω/A]

a = 0.101884 1/V; b = 2.583663 1/A

Standard deviation of the slope and the intercept

s_r residual standard deviation

$$s_r^2 = \frac{\sum(a \cdot x_i + b - y_i)^2}{N-2}$$

$$\mathrm{Var}[a] = \frac{s_r^2}{N \cdot \left(\overline{x^2} - \bar{x}^2\right)} \quad \rightarrow \quad s_a = \sqrt{\mathrm{Var}[a]}$$

$$\mathrm{Var}[b] = \overline{x^2} \cdot \mathrm{Var}[a] \quad \rightarrow \quad s_b = \sqrt{\mathrm{Var}[b]}\,.$$

Excel:

$(a \cdot x_i + b - y_i)^2$: $\left(a \cdot R_v + b - \frac{1}{I}\right)^2$ $\rightarrow$ sum $\rightarrow$ s_r^2 = ...

s_r^2 = 1.6137 1/A^2

Var[a] = 1.061E-05 1/V^2 s_a = 0.00326 1/V

Var[b] = 1.2776902 1/A^2 s_b = 1.13035 1/A

a = (0.101884 ± 0.00326) 1/V

b = (2.583663 ± 1.13035) 1/A

Error interval

E and R_i are calculated from a and b.

a = (0.101884 ± 0.00326) 1/V

b = (2.583663 ± 1.13035) 1/A

$$a = \frac{1}{E} \quad \longrightarrow \quad E = \frac{1}{a}$$

$$\Delta E = \sqrt{\left(\frac{\partial E}{\partial a} \cdot \Delta a\right)^2 + \left(\frac{\partial E}{\partial b} \cdot \Delta b\right)^2} = \sqrt{\left(-\frac{1}{a^2} \cdot \Delta a\right)^2 + 0} = \left|\frac{1}{a^2} \cdot \Delta a\right| = 0.3138 \text{ V}$$

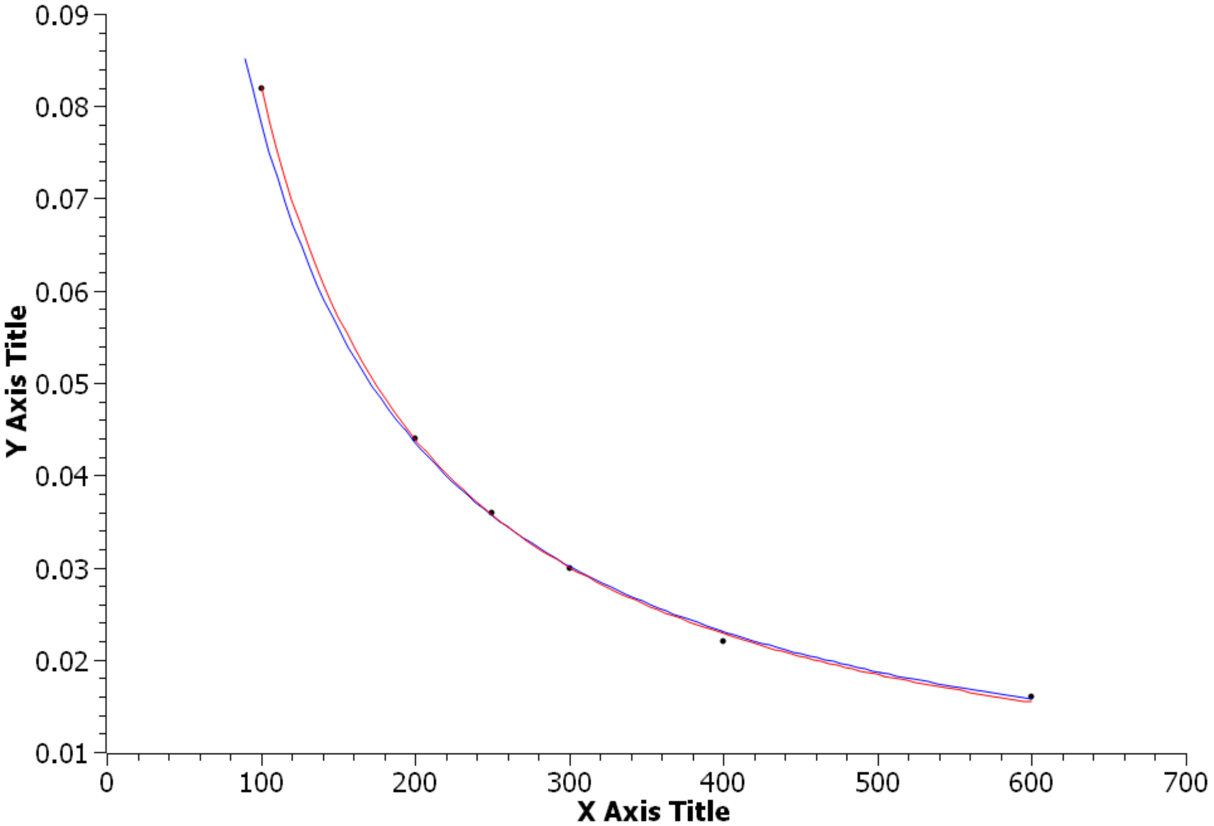
$$b = \frac{R_i}{E} \quad \longrightarrow \quad R_i = \frac{b}{a}$$

$$\Delta R_i = \sqrt{\left(\frac{\partial R_i}{\partial a} \cdot \Delta a\right)^2 + \left(\frac{\partial R_i}{\partial b} \cdot \Delta b\right)^2} = \sqrt{\left(-\frac{b}{a^2} \cdot \Delta a\right)^2 + \left(\frac{1}{a} \cdot \Delta b\right)^2} = 11.124\ \Omega$$

E = (9.82 $\pm$ 0.31) V

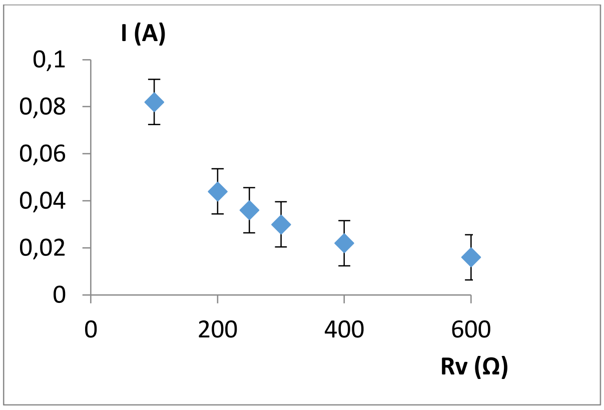
R_b = (25.4 $\pm$ 11.1) Ω

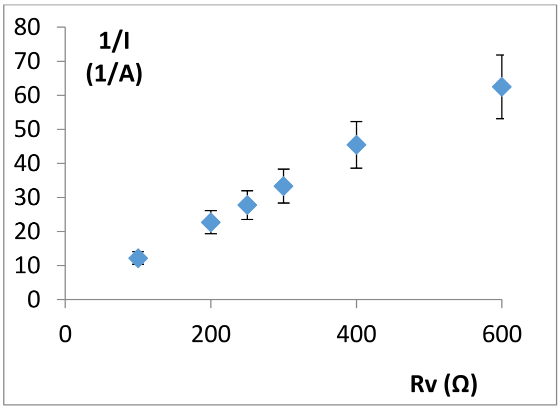
SUMMARY



hiperbola
(SciDAVis)
E = (9.48 +/- 0.132) V
R_i = (15.6 +/- 1.97) Ω
R^2 = 0.9996

linearization
(Excel)
E = (9.82 ± 0.31) V
R_i = (25 ± 11.1) Ω
R^2 = 0.9959

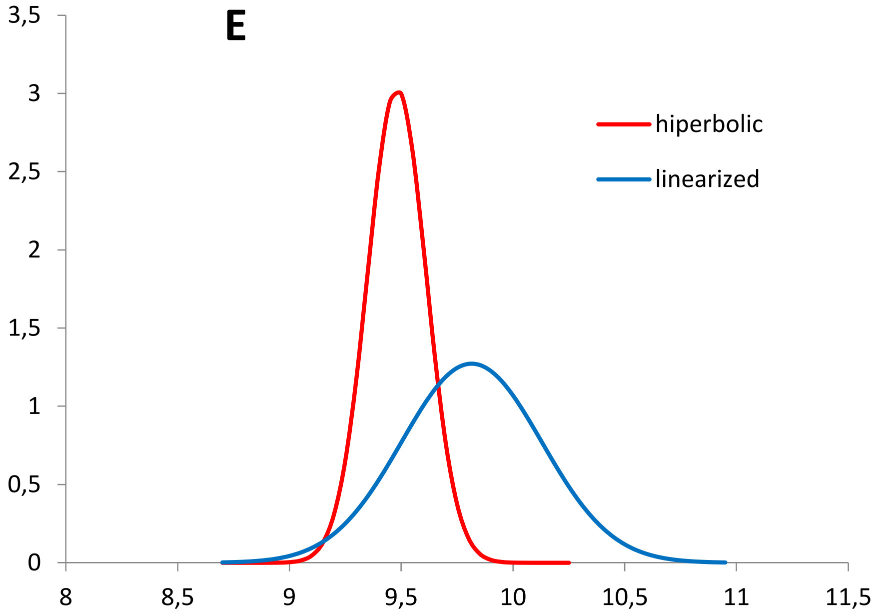




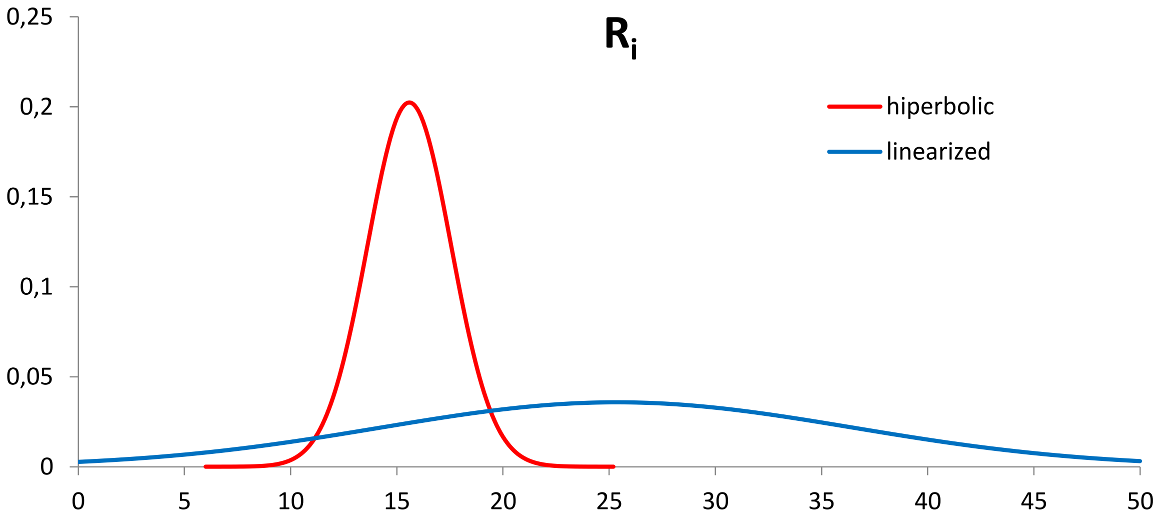
Normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\; e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

E electromotive force: E = (9.48 +/- 0.132) V   E = (9.82 ± 0.31) V



R_i internal resistance: R_i = (15.6 +/- 1.97) Ω R_i = (25 ± 11.1) Ω



In case of a **linear function** *y = a x* (the intercept of the line is zero)

$S(a) = \sum_{i=1}^{N}(ax_i - y_i)^2 \rightarrow min.$ leads to $a = \frac{\overline{x \cdot y}}{\overline{x^2}}$.

and the **standard deviation** of the slope can be calculated applying the following formula:

$$s_a = \sqrt{\frac{\sum_{i=1}^{N}(a \cdot x_i - y_i)^2}{N \cdot (N-1) \cdot \overline{x^2}}}$$