

Resonance in a series RLC circuit

THEORY

1. Alternating current and voltage

An alternating current (AC) is an electrical current where the magnitude and direction of the current varies cyclically, so as the time average is zero (as opposed to direct current (DC), where the direction of the current stays constant). The usual waveform of an AC power circuit is a sine wave. (However, in certain applications different waveforms are used, such as triangular or square waves.)

The generators in the power plants produce sinusoidal voltages. The current flowing through the electric appliances using AC power supply is also sinusoidal. The AC voltages and currents are characterized by their

- frequency f [Hz], or angular frequency $\omega = 2\pi f = 2\pi/T$ [s^{-1}],
- amplitude U_0 [V] and I_0 [A], respectively, and
- phase constant φ_U and φ_I [–], respectively:

$$U(t) = U_0 \cos(\omega t + \varphi_U) \quad \text{and} \quad I(t) = I_0 \cos(\omega t + \varphi_I).$$

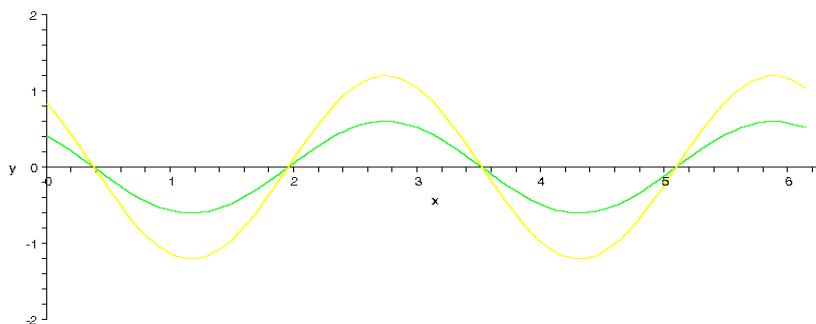
The voltage and the current are not necessarily in-phase, there can be some phase difference $\Delta\varphi = \varphi_U - \varphi_I$ between them (φ_U is first). Choosing the phase constant of the current zero, the time dependence of the current and voltage in an AC circuit can be written as

$$I(t) = I_0 \cos(\omega t) \quad \text{and} \quad U(t) = U_0 \cos(\omega t + \Delta\varphi).$$

2. Resistor in an AC circuit

Ohm's Law is valid for the time-dependent voltage and current so

$$U_R(t) = R I(t). \tag{1}$$



green: $I(t)$, yellow: $U_R(t)$

Across a resistor the voltage and the current are in-phase ($\Delta\varphi = 0$),
and for the amplitudes holds $U_{R0} / I_{R0} = R$.

3. Capacitor in an AC circuit

The voltage across the capacitor is proportional to the charge Q accumulated on its plates:

$$U_C(t) = \frac{1}{C} Q(t) \quad \text{where } C \text{ is the capacitance in Farad [F].}$$

As the current is the derivative of the charge $I(t) = \frac{dQ(t)}{dt}$,

so the integral of the current is proportional to the voltage:

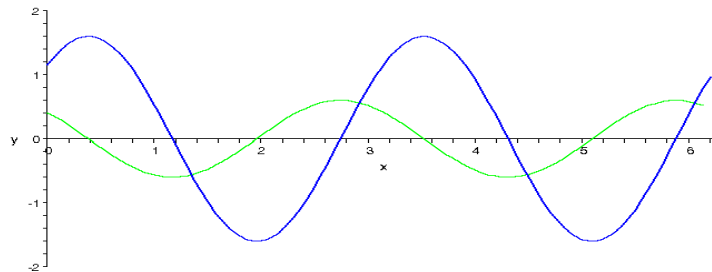
$$U_C(t) = \frac{1}{C} \int I(t) dt.$$

In case of a sinusoidal current $I(t) = I_0 \cos(\omega t)$ the voltage is

$$U_C(t) = \frac{1}{C} \int I_{C0} \cos(\omega t) dt = \frac{1}{\omega C} I_{C0} \sin(\omega t) = \frac{1}{\omega C} I_{C0} \cos(\omega t - \frac{\pi}{2}) .$$

There will be a phase shift between the voltage and the current: $\Delta\varphi = -\pi/2$,

and $U_{C0} = \frac{1}{\omega C} I_{C0}$, i.e. $\frac{U_{C0}}{I_{C0}} = \frac{1}{\omega C}$. (2)



green: $I(t)$, blue: $U_C(t)$

Across a capacitor the phase difference is $\Delta\varphi = -\pi/2$ (the voltage is delayed),

and for the amplitudes holds $U_{C0} / I_{C0} = \frac{1}{\omega C}$.

4. Inductor in an AC circuit

In case of a time-dependent current a strong magnetic field is generated inside the coil due to Ampere's Law, and due to the time-varying magnetic field inside the coil a voltage is induced, according to Faraday's Law of electromagnetic induction (which by Lenz's Law opposes the change in current that created it). So the voltage across the coil is proportional to the derivative of the current flowing through it:

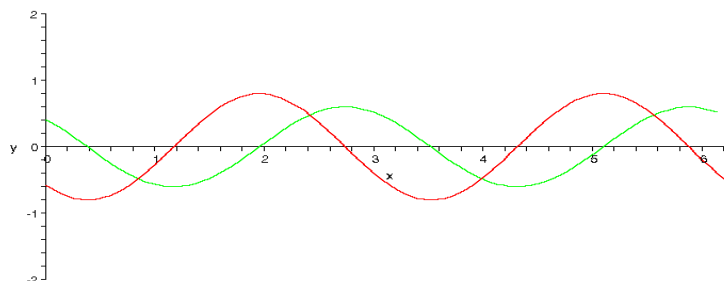
$$U_L(t) = L \frac{dI(t)}{dt} \quad \text{where } L \text{ is the inductance in Henry [H].}$$

In case of a sinusoidal current $I(t) = I_0 \cos(\omega t)$ the voltage is

$$U_L(t) = L \frac{d(I_0 \cos(\omega t))}{dt} = -L\omega I_{L0} \sin(\omega t) = L\omega I_{L0} \cos(\omega t + \frac{\pi}{2}) .$$

There will be a phase shift between the voltage and the current: $\Delta\varphi = +\pi/2$,

and $U_{L0} = L\omega I_{L0}$, i.e. $\frac{U_{L0}}{I_{L0}} = L\omega$. (3)



green: $I(t)$, red: $U_L(t)$

Across a coil the phase difference is $\Delta\varphi = +\pi/2$ (the current is delayed),

and for the amplitudes holds $U_{L0} / I_{L0} = \omega L$.

5. Using complex quantities for AC circuits

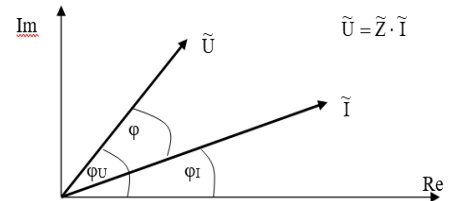
Instead of using the sine and cosine functions, it is simpler to treat AC circuits if we represent AC currents and voltages by complex quantities:

$$\tilde{I} = I_0 e^{i\omega t} = I_0 \cos(\omega t) + I_0 \sin(\omega t) i ,$$

$$\tilde{U} = U_0 e^{i(\omega t + \Delta\varphi)} = U_0 \cos(\omega t + \Delta\varphi) + U_0 \sin(\omega t + \Delta\varphi) i .$$

The real time-dependent current or voltage is obtained as the real part of the complex form. The ratio of the real time-dependent voltage and current would change with time in an AC circuit, but it is constant in the complex representation. The ratio of the complex voltage to the complex current is called **complex impedance** and denoted by $\tilde{Z} = Z e^{i\varphi}$:

$$\tilde{Z} = \frac{\tilde{U}}{\tilde{I}} = \frac{U_0 e^{i(\omega t + \Delta\varphi)}}{I_0 e^{i\omega t}} = \frac{U_0}{I_0} e^{i\Delta\varphi} .$$



The magnitude of the complex impedance

$$Z = |\tilde{Z}| = \frac{U_0}{I_0}$$

is equal to the ratio of the amplitude of the voltage to the amplitude of the current, and

the phase of the complex impedance φ

is equal to the phase difference between voltage and current,

$$\varphi = \Delta\varphi = \varphi_U - \varphi_I .$$

6. Complex impedance of resistors, capacitors and inductors

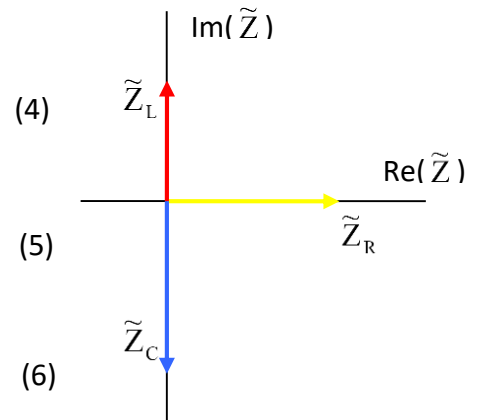
The impedance of a resistor is equal to the resistance: $\tilde{Z}_R = R$.
There is no phase difference between the voltage and the current.

The impedance of a capacitor is $\tilde{Z}_C = \frac{1}{\omega C i} = -\frac{1}{\omega C} i$.

The voltage of a capacitor lags behind the current by $\pi/2$.

The impedance of an inductor is $\tilde{Z}_L = \omega L i$.

The voltage of an inductor leads the current by $\pi/2$.



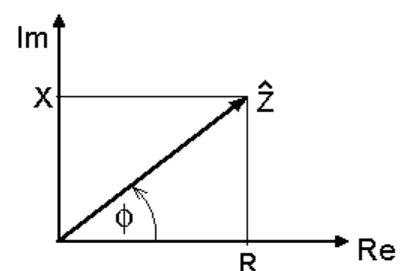
7. Resultant impedance

Making a two-pole by connecting resistors, capacitors and inductors in any way, it can be replaced by a resultant impedance. Though the impedance of the constructing elements is either real or imaginary, the resultant impedance in a general case has both a real and an imaginary part: $\tilde{Z} = R + X i$.

We call the real part "resistance", R ;
and the imaginary part "reactance", X .

The magnitude Z and phase φ of the impedance are related to R and X as follows:

$$Z = \sqrt{R^2 + X^2}, \quad \text{and} \quad \tan\varphi = \frac{X}{R} .$$



The resultant impedance is calculated similarly to the resultant resistance:

in a series connection $\tilde{Z}_{res, series} = \tilde{Z}_1 + \tilde{Z}_2 + \dots$,

in a parallel connection $\frac{1}{\tilde{Z}_{res, parallel}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \dots$.

According to the resultant complex impedance the real AC voltage and current are related so that

- the ratio of the amplitudes (or rms or effective values) of the voltage to the current is given by the magnitude of the impedance, and
- the phase difference between the voltage and the current is equal to the phase of the impedance.

The AC **power** dissipated on an impedance is

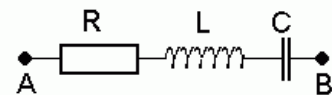
$$P = U_{rms} I_{rms} \cos\varphi = Z I_{rms}^2 \cos\varphi .$$

On an inductor or capacitor, the phase is $\varphi = \pm \pi/2$, and the power is zero.

Only the resistors dissipate power.

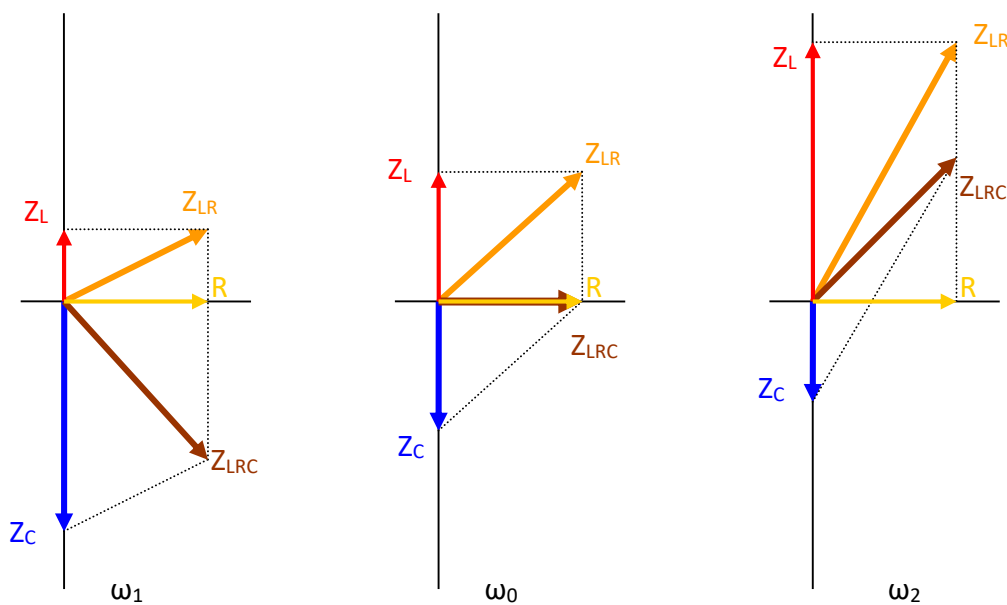
In the following we shall focus on the simplest possible circuit, the series RLC circuit.

8. Series RLC circuit



The complex impedance of a series RLC circuit at angular frequency ω is

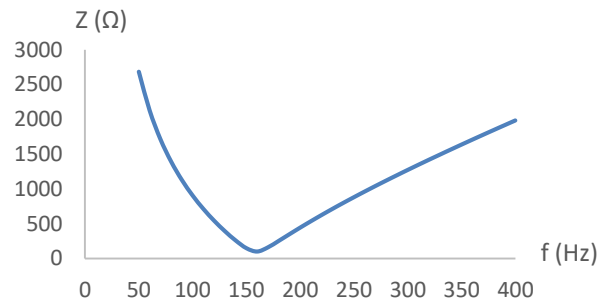
$$\tilde{Z}_{RLC} = \tilde{Z}_R + \tilde{Z}_L + \tilde{Z}_C = R + \omega L i - \frac{1}{\omega C} i = R + \left(\omega L - \frac{1}{\omega C} \right) i . \tag{7}$$



Complex impedance of a capacitor, real inductor and their resultant as the angular frequency increases, $\omega_1 < \omega_0 < \omega_2$.

The magnitude of the impedance depends on the angular frequency:

$$Z_{RLC}(\omega) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} . \quad (8)$$



As we know, the magnitude of the complex impedance is equal to the ratio of the amplitude of the voltage to that of the current, or – instead of the amplitude – the rms or effective values can be also used:

$$Z_{RLC} = |\tilde{Z}_{RLC}| = \frac{U_{0,RLC}}{I_0} = \frac{U_{rms,RLC}}{I_{rms}} = \frac{U_{eff,RLC}}{I_{eff}} . \quad (9)$$

In a series RLC circuit

$$\frac{U_{rms,RLC}}{I_{rms}} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} . \quad (10)$$

The magnitude of the impedance takes its minimal value at $\omega = \omega_0$ when

$$\omega_0 L = \frac{1}{\omega_0 C} \quad \rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}} .$$

The frequency where this holds is called the **resonant frequency** f_0 of the RLC circuit:

$$f_0 = \frac{1}{2\pi \sqrt{LC}} \quad \text{Thomson's formula} \quad (11)$$

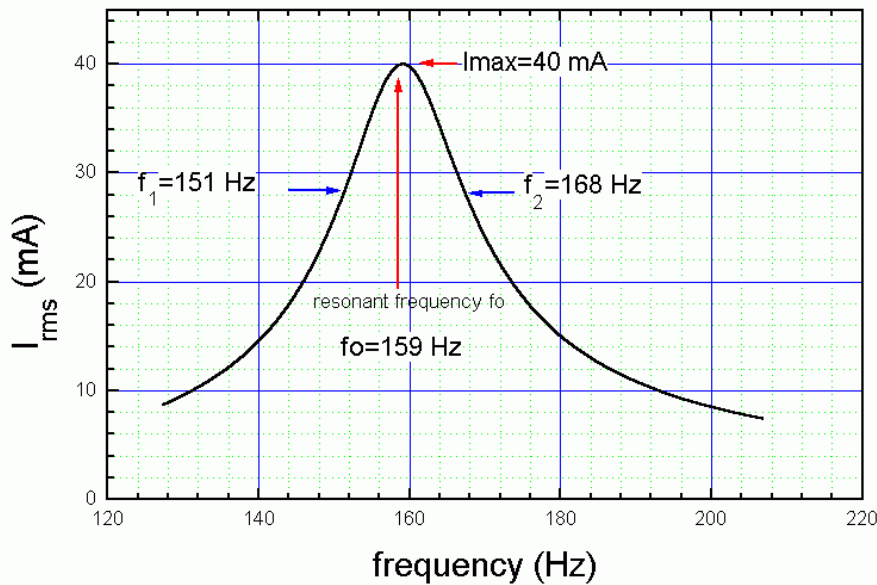
At the resonant frequency $Z_{RLC}(f_0) = R$. (12)

Driving the RLC circuit with a constant-rms value AC voltage ($U_{rms,RLC} = \text{const.}$) and increasing the angular frequency, the current will increase first, reach a maximum and then decrease, according to the function

$$I_{rms} = \frac{U_{rms,RLC}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} . \quad (13)$$

The maximum current is $I_{rms} = \frac{U_{rms,RLC}}{R}$ at the resonant frequency f_0 . (14)

Such a resonance curve is shown in the next figure.



Here $U_{rms} = 4.0 \text{ V}$.

What data of the circuit can be obtained from such a resonant curve? We can read

- the maximum current I_{max} ; here $I_{max} = 40 \text{ mA}$;
- the resonant frequency f_0 ; here $f_0 = 159 \text{ Hz}$;
- f_1 and f_2 , the frequencies where the current is $\frac{I_{max}}{\sqrt{2}}$; here $f_1 = 151 \text{ Hz}$, $f_2 = 168 \text{ Hz}$.

From these data we can get

- the resistance: $R = \frac{U_{rms}}{I_{rms}}$; here $R = 4.0/40 \cdot 10^{-3} = 100 \Omega$;
- the so-called quality factor: $Q = \frac{f_0}{f_2 - f_1}$; (15)
here $Q = 159/(168-151) = 9.353$.

The **Q quality factor** characterizes the sharpness of the resonance. It can be also calculated as

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} \quad \text{or} \quad Q = \frac{\omega_0 L}{R} \quad \text{or} \quad Q = \frac{1}{\omega_0 C R} , \quad (16)$$

as it is equal to the voltage across a reactive element at the resonant frequency divided by the voltage across the whole circuit at the resonant frequency.

One possibility to calculate the inductance L and the capacitance C is to use these formulas and substitute the values of R , Q and $\omega_0 = 2\pi f_0$:

$$\text{here } L = \frac{QR}{2\pi f_0} = 0.9359 \text{ H}, \text{ and } C = \frac{1}{2\pi f_0 QR} = 1.070 \mu\text{F}.$$

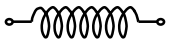
Note that one of them could be also calculated from the resonant frequency as $LC = \frac{1}{(2\pi f_0)^2}$.

L and C can be also calculated by measuring the current, and the voltage separately across the inductor and the coil, and use the formulas of the impedances.


MEASUREMENT

Tools:

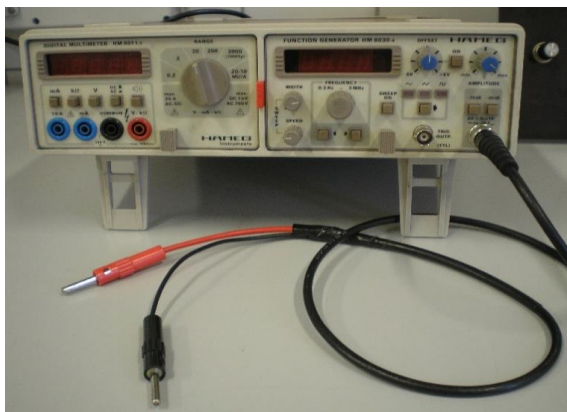


Inductor  L, R



Capacitor  C

The inductor – a solenoid equipped with an iron core – has its own resistance in addition to the inductance. It is considered as an ideal inductor and a resistor connected in series.

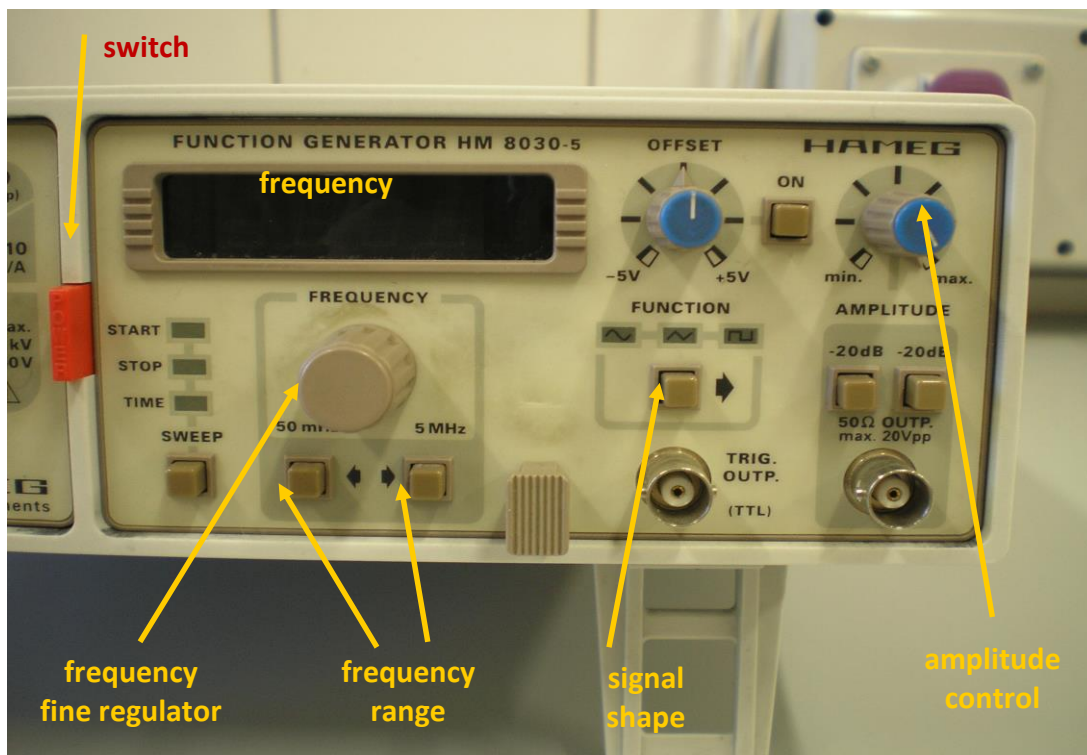



On the left: ammeter.

On the right: AC generator with cable.

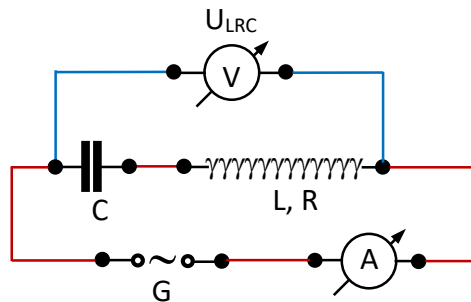


Voltmeter



AC generator  G

1. Resonance curve



- Set up the series RLC circuit using the capacitor and inductor.
Connect the circuit to the AC generator through the ammeter.
- Connect the voltmeter to measure the voltage U_{LRC} across the RLC circuit, i.e. across the capacitor and inductor.
- Switch on the generator and the meters, and without regulating the voltage find the approximate frequency where the current is maximal.
- Keeping the voltage constant, find the frequency where the current is maximal: this is the resonant frequency f_0 .
- Measure the current 1, 2, 5, 10, 20, 30, ... Hz away from the resonant frequency till the current drops about to half of the maximum value.

EVALUATION

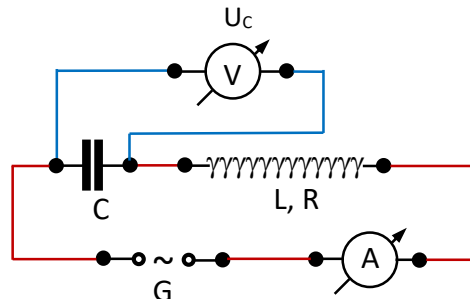
- Plot the $I(f)$ curve.
- Read the resonant frequency f_0 from the curve.
- Calculate the ohmic resistance of the inductor as

$$R = U_{\text{gen}} / I_{\text{max}} .$$
- Read the frequencies f_1 and f_2 where the current is $\frac{I_{\text{max}}}{\sqrt{2}}$, and calculate the quality factor as

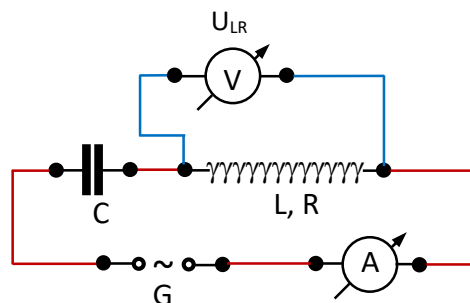
$$Q = f_0 / (f_2 - f_1).$$

2. Vector diagrams

- Set the resonant frequency f_0 and read the current and the voltage across the whole circuit U_{LRC} .
- Disconnect the voltmeter and connect it to measure the voltage across the capacitor U_C :



- Disconnect the voltmeter and connect it to measure the voltage across the inductor U_{LR} :



- Repeat this measurement at the frequencies $0.9 f_0$ and $1.1 f_0$.

EVALUATION

2A. The inductance L and the capacitance C can be calculated from the measured current and voltages. After calculating these values, we check that our measurements and calculations were correct by calculating f_0 and Q again, and comparing it to the values obtained in task 1.

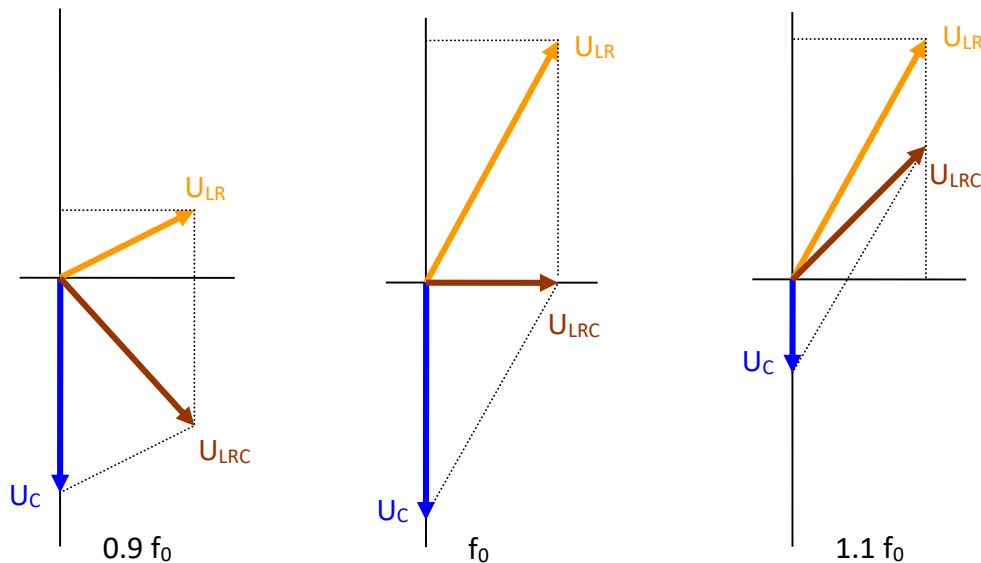
The following calculations can be performed with the data sets measured at any of the three frequencies ($0.9f_0$ or f_0 or $1.1f_0$).

- Calculate the angular frequency $\omega = 2\pi f$.
- Calculate the capacitance from the relation $\frac{U_C}{I} = Z_C = \frac{1}{\omega C}$.
- Calculate the inductance from the relation $\frac{U_{LR}}{I} = Z_{LR} = \sqrt{R^2 + (\omega L)^2}$. Use the value of R that was calculated in task 1.
- Calculate the resonant frequency using the Thompson formula (11) and compare it with the measured value.
- Calculate the quality factor using the formula

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} \quad \text{and compare it with the value obtained from the } I(f) \text{ curve.}$$

2B. Construct the vector diagrams for the complex voltages at the angular frequencies measured: $0.9 f_0$, f_0 , and $1.1 f_0$.

This is how they should look like:



(The diagrams are not proportional.)

We know that $\tilde{U}_C + \tilde{U}_{LR} = \tilde{U}_{LRC}$. The difficulty is that only the length of these vectors was measured. We know that \tilde{U}_C points downwards as the capacitor is ideal, but we do not know the direction of \tilde{U}_{LR} and \tilde{U}_{LRC} because we do not know their phase. So the vector diagrams can be constructed by first constructing the lower triangle of the parallelogram using a compass.

First draw \tilde{U}_C , then draw an arc around the origin with a radius of U_{LRC} and another arc around the tip of \tilde{U}_C with a radius of U_{LR} . This way you get the tip of \tilde{U}_{LRC} . \tilde{U}_{LR} is obtained by shifting the other edge to the origin. Please add a proper scale in ... V / cm.

ENTRANCE TEST

Describe the measurements: tools, instructions, quantities measured and calculated, formulas.

True or false? Justify your answer.

In a series LRC circuit

- the voltage across the inductor lags the current;
- the voltage across the capacitor lags the current;
- the voltage across the inductor leads the voltage across the capacitor;
- the voltage across the inductor lags the voltage across the capacitor;
- the generator voltage leads the voltage across the capacitor;
- the generator voltage leads the voltage across the inductor.

In a series LRC circuit at the resonant frequency

- the resultant impedance is minimal;
- the resultant impedance is maximal;
- the impedance of the capacitor is minimal;
- the impedance of the inductor is minimal;
- the current is minimal if $U_{\text{gen}} = \text{const.}$;
- the current is maximal if $U_{\text{gen}} = \text{const.}$;
- the voltage across the capacitor can be larger than the generator voltage;
- the voltage across the inductor can be larger than the generator voltage;
- the voltage across the resistor can be larger than the generator voltage;
- the voltage across the capacitor can be larger than the voltage across the ideal inductor.

Problems

1. We set up a series RLC circuit using a real inductor that has both an inductance and a resistance, and a capacitor, and connect it to an AC generator. $L = 0.5 \text{ H}$; $R = 480 \Omega$; $C = 200 \text{ nF}$; $U_{\text{gen}} = 7.0 \text{ V}$.

- a) Calculate the current at $f = 600 \text{ Hz}$.
- b) Calculate the voltage across the capacitor at $f = 600 \text{ Hz}$.
- c) Calculate the voltage across the real inductor at $f = 600 \text{ Hz}$.
- d) Determine the resonant frequency of the circuit.
- e) Calculate the current at the resonant frequency.
- f) Calculate the voltage across the capacitor at the resonant frequency.
- g) Calculate the voltage across the real inductor at the resonant frequency.
- h) Determine the quality factor of the circuit.

Solution

a) $U_{\text{gen}} = U_{\text{LRC}} = I Z_{\text{LRC}} \rightarrow I = U_{\text{gen}} / Z_{\text{LRC}} .$

$$Z_{\text{LRC}} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{480^2 + \left(2\pi \cdot 600 \cdot 0.5 - \frac{1}{2\pi \cdot 600 \cdot 200 \cdot 10^{-9}}\right)^2} = 736.6 \, \Omega ,$$

$$I = U_{\text{gen}} / Z_{\text{LRC}} = 7.0 / 736.6 = 0.009504 \, \text{A} = 9.504 \, \text{mA} .$$

b) $U_C = I Z_C , \quad Z_C = |\tilde{Z}_C| = \left| -\frac{1}{\omega C} i \right| = \frac{1}{\omega C} .$

$$Z_C = \frac{1}{2\pi \cdot 600 \cdot 200 \cdot 10^{-9}} = 1885 \, \Omega ,$$

$$U_C = 0.009504 \cdot 1885 = 17.91 \, \text{V} .$$

c) $U_{\text{LR}} = I Z_{\text{LR}} , \quad Z_{\text{LR}} = |\tilde{Z}_{\text{LR}}| = |R + \omega L i| = \sqrt{R^2 + (\omega L)^2} .$

$$Z_{\text{LR}} = \sqrt{480^2 + (2\pi \cdot 600 \cdot 0.5)^2} = 1945 \, \Omega ,$$

$$U_{\text{LR}} = 0.009504 \cdot 1945 = 18.49 \, \text{V} .$$

Note that both $U_C > U_{\text{LRC}}$ and $U_{\text{LR}} > U_{\text{LRC}}$, and $U_C + U_{\text{LR}} \neq U_{\text{LRC}}$.

d) Thomson' formula

$$f_0 = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{0.5 \cdot 200 \cdot 10^{-9}}} = 503.3 \, \text{Hz} .$$

e) At the resonant frequency $Z_{\text{LRC}}(f_0) = R = 480 \, \Omega$,

$$I(f_0) = U_{\text{LRC}} / R = 7.0 / 480 = 0.01458 \, \text{A} = 14.58 \, \text{mA} \quad (\text{this is the maximal current in this circuit})$$

f) $Z_C = \frac{1}{2\pi \cdot 503.3 \cdot 200 \cdot 10^{-9}} = 1581 \, \Omega$,

$$U_C = 0.009504 \cdot 1581 = 23.06 \, \text{V} .$$

g) $Z_{\text{LR}} = \sqrt{480^2 + (2\pi \cdot 503.3 \cdot 0.5)^2} = \sqrt{480^2 + 1581^2} = 1652 \, \Omega$,

$$U_{\text{LR}} = 0.009504 \cdot 1652 = 24.10 \, \text{V} .$$

Note that both $U_C > U_{\text{LRC}}$ and $U_{\text{LR}} > U_{\text{LRC}}$, and $U_C + U_{\text{LR}} \neq U_{\text{LRC}}$.

h) $Q = \frac{1}{R} \sqrt{\frac{L}{C}} = \frac{1}{480} \sqrt{\frac{0.5}{200 \cdot 10^{-9}}} = 3.294$

2. We set up a series RLC circuit using a real inductor that has an inductance $L = 2.5 \, \text{H}$ and a resistance $R = 250 \, \Omega$, and a capacitor with unknown capacitance, and connect it to an AC generator. The resonant frequency of the circuit is $f_0 = 318 \, \text{Hz}$.

a) Determine the unknown capacitance.

b) Calculate the ratio U_{gen} / U_C at the resonant frequency.

Solution

a) Thomson' formula:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \rightarrow C = \frac{1}{(2\pi f_0)^2 L} = \frac{1}{(2\pi \cdot 318)^2 \cdot 2.5} = 1.002 \cdot 10^{-7} \text{ F} = 100.2 \text{ nF} .$$

b) $U_{\text{gen}} / U_C = U_{\text{LRC}} / U_C = (I Z_{\text{LRC}}) / (I Z_C) = Z_{\text{LRC}} / Z_C$

$Z_C = 1 / (2\pi f C)$, and at the resonant frequency $Z_{\text{LRC}}(f_0) = R$, so

$$U_{\text{gen}}(f_0) / U_C(f_0) = R / \{1/(2\pi f_0 C)\} = R \cdot (2\pi f_0 C) = 250 \cdot (2\pi \cdot 318 \cdot 1.002 \cdot 10^{-7}) = 0.05005 = 5.005\% ,$$

i.e. $U_{\text{gen}} \ll U_C$!

3. The effective value of the current was measured in a series RLC circuit while increasing the angular frequency.

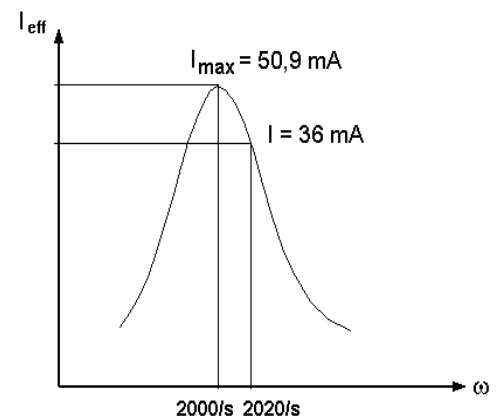
The generator voltage was constant, $U_{\text{eff,gen}} = 45.81 \text{ V}$.

The current is maximal at $\omega_0 = 2000 \text{ 1/s}$: $I_{\text{eff,max}} = 50.9 \text{ mA}$.

We also know that at $\omega_1 = 2020 \text{ 1/s}$ the measured current

is $I_{\text{eff,1}} = 36.0 \text{ mA}$.

Determine the resistance R , the inductance L and the capacitance C .



Solution

The resultant impedance $U_{\text{eff}} / I_{\text{eff}} = Z$ is a function of the angular frequency:

$$Z(\omega) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} .$$

The resonant angular frequency is $\omega_0 = 2000 \text{ 1/s}$.

At this frequency

$$Z(\omega_0) = R, \text{ i.e. } U_{\text{eff}} / I_{\text{eff,max}} = R.$$

Substitution:

$$R = 45.81 \text{ V} / 50.9 \cdot 10^{-3} \text{ A} = 900 \ \Omega.$$

We can also write that

$$\omega_0 L = \frac{1}{\omega_0 C} : \quad 2000 L = \frac{1}{2000 C} .$$

Furthermore, the resultant impedance can be calculated at $\omega_1 = 2020 \text{ 1/s}$:

$$Z(\omega_1) = U_{\text{eff}} / I_{\text{eff,1}} = 45.81 \text{ V} / 36.0 \cdot 10^{-3} \text{ A} = 1272.5 \ \Omega,$$

so

$$Z(\omega_1) = \sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2} : \quad 1272.5 = \sqrt{900^2 + \left(2020 L - \frac{1}{2020 C}\right)^2}.$$

The solution of the equation system

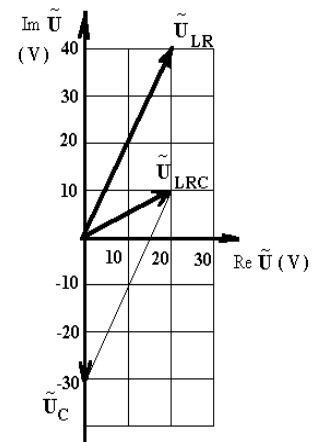
$$2000 L = \frac{1}{2000 C}$$

$$1272.5 = \sqrt{900^2 + \left(2020 L - \frac{1}{2020 C}\right)^2}$$

gives $L = 22.60 \text{ H}$ and $C = 1.106 \cdot 10^{-8} \text{ F} = 11.06 \text{ nF}$.

4. We set up a series RLC circuit using a real inductor that has both an inductance and a resistance, and a capacitor, and connect it to an AC generator. The angular frequency is set to $\omega = 5000 \text{ 1/s}$, and the current to $I = 5 \text{ mA}$. We measured the voltage across the real inductor U_{LR} , across the capacitor U_C , and the generator voltage $U_{\text{gen}} = U_{LRC}$, and constructed the vector diagram on the complex plane.

- Determine the resistance R , the inductance L and the capacitance C .
- Determine the resonant frequency and the quality factor of the circuit.



Solution

a) The complex voltages can be read from the vector diagram:

$$\tilde{U}_C = -30 i \text{ [V]}$$

$$\tilde{U}_{LR} = 20 + 40 i \text{ [V]}$$

and $\tilde{U}_{LRC} = 20 + 10 i$, which is not needed for the calculations, but we can see that this is the sum of the two voltages: $\tilde{U}_C + \tilde{U}_{LR} = -30 i + (20 + 40 i) = 20 + 10 i \text{ [V]}$

The current is $I = 5 \text{ mA} = 0.005 \text{ A}$, so the complex impedances are

$$\tilde{Z}_C = \tilde{U}_C / I = -30 i / 0.005 = -6000 i \text{ [\Omega]}$$

$$\tilde{Z}_{LR} = \tilde{U}_{LR} / I = (20 + 40 i) / 0.005 = 4000 + 8000 i \text{ [\Omega]}$$

$$\text{and } \tilde{Z}_{LRC} = \tilde{U}_{LRC} / I = (20 + 10 i) / 0.005 = 4000 + 2000 i \text{ [\Omega]}$$

We know that

$$\tilde{Z}_C = \frac{1}{\omega C i} = -\frac{1}{\omega C} i : \quad -6000 i = -\frac{1}{5000 C} i \quad \rightarrow \quad C = 3.333 \cdot 10^{-8} \text{ F} = 33.33 \text{ nF};$$

$$\tilde{Z}_{LR} = R + \omega L i : \quad 4000 + 8000 i = R + 5000 L i \quad \rightarrow \quad R = 4000 \text{ }\Omega, \quad L = 1.6 \text{ H}.$$

b) Thomson's formula:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{1.6 \cdot 3.333 \cdot 10^{-8}}} = 689.2 \text{ Hz.}$$

Quality factor:

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} = \frac{1}{4000} \sqrt{\frac{1.6}{3.333 \cdot 10^{-8}}} = 1.732 .$$

5. We set up a series RLC circuit using a real inductor that has both an inductance and a resistance, and a capacitor, and connect it to an AC generator. At the resonant frequency we measured the generator voltage: $U_{\text{gen}} = U_{\text{LRC}} = 4.8 \text{ V}$, and the voltage across the capacitor: $U_C = 3.6 \text{ V}$. Determine the voltage across the real inductor U_{LR} .

Solution

Let us write the voltages in complex form.

We know that the capacitor voltage is a negative imaginary value: $\tilde{U}_C = -3.6 i \text{ [V]}$.

At the resonant frequency $Z_{\text{LRC}}(f_0) = R$, this means that the imaginary part of the generator voltage is zero, $\tilde{U}_{\text{LRC}} = 4.8 \text{ V}$.

$$\tilde{U}_{\text{LRC}} = \tilde{U}_C + \tilde{U}_{\text{LR}} \quad \rightarrow$$

$$\tilde{U}_{\text{LR}} = \tilde{U}_{\text{LRC}} - \tilde{U}_C = 4.8 - (-3.6 i) = 4.8 + 3.6 i \text{ [V]}.$$

The measured quantity is the absolute value of the complex voltage:

$$|\tilde{U}_{\text{LR}}| = \sqrt{4.8^2 + 3.6^2} = 6 \text{ V} .$$

Note that $U_{\text{LR}} = U_{\text{LRC}} - U_C = 4.8 - 3.6 = 1.2 \text{ V}$ is not true because of the phase shift! The measured values are the effective values, proportional to the amplitudes, but the maximum values of the time-dependent voltages are shifted and so the amplitudes do not add up.

