Errors of measurement data

A quantity – unless it is countable – cannot be given with absolute accuracy. The value given as the measure of the quantity will be always different from the real value. The deviation of the given value from the real one is the error.

The most inherent error comes from the representation of numbers: we can give a number only with finite number of digits, and read the measurement results from a meter also with a finite number of digits. This **reading error** cannot be avoided. We take 1 unit of the last significant digit of a measurement or of a given quantity as the error if the error is not otherwise specified. Measuring a quantity, the measured value can deviate from the real one more than the reading error. The magnitude of the deviation is called the error of the measured value.

The **absolute error Δx** of a quantity x is

$$\Delta x = |x_{measured} - x_{real}|.$$ (1)

The **relative error δx** of the same quantity is

$$\delta x = \Delta x / x_{real}.$$ (2)

The relative error is usually given in percents.

The error can be caused by a faulty meter. To avoid this **systematic error** caused by the measuring apparatus, all meters have to be checked regularly and be calibrated against some known standard.

Even with a well calibrated meter and a careful measurement procedure we might get different results if we repeat the measurement. When the fluctuations of measurement results are caused by random effects we speak about **accidental errors**. Such random effects can arise from the measured quantity which might not have the same value at every instant, or from the measuring apparatus or from the observer. The accidental errors obey the laws of probability theory. The possible result of the measurement is a random variable with a probability distribution. Usually it is supposed a Gaussian distribution, with probability density of

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$ (3)

μ is the expectation value (mean) of the distribution, and
σ is the standard deviation, the "half-width" of the probability density curve.

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1. Confidence interval

We assume that the measurement results fluctuate around the "real value" μ, and we estimate it by the **arithmetic mean** $\bar{x}$ of <u>all the data</u> obtained when we repeated the measurement N times:

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \; .$$
(4)

The fluctuation of the measurement results is characterized by the standard deviation σ, estimated by the **experimental standard deviation s_x :**

$$s_x = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1}} \; .$$
(5)

This standard deviation characterizes both the measurement procedure and tool and also the ensemble from where a sample was taken.

It is also important to know, how much our measurement result $\bar{x}$ might deviate from the "real value". The probable deviation of $\bar{x}$ from the "real value" is called the **standard deviation of the mean $s_{\bar{x}}$** , estimated from the data of the repeated measurement as

$$s_{\bar{x}} = \frac{s_x}{\sqrt{N}} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N(N-1)}} \; .$$
(6)

The standard deviation of the mean decreases with the number of measurements. We can make the measurement more precise if we repeat it and calculate the mean of the results.

The interval around the given value that contains the "real value" with P probability is called "**confidence interval**" or "**error interval**", and **P is the "confidence level"**. We get the radius of this interval by using the so called **Student's t parameter**, given for a specified level of confidence P and number of repeated measurements N. The parameter t can be found in tables for Student's distribution.

Values of t (Student's parameter)
 at N number of repeated measurement data and at P confidence level

P \ N	0.8	0.9	0.95	0.975	0.99	0.995
2	3.078	6.314	12.706	25.452	63.657	127.32
3	1.886	2.920	4.303	6.205	9.925	14.089
4	1.638	2.353	3.182	4.176	5.841	7.453
5	1.553	2.132	2.776	3.495	4.604	5.598
6	1.476	2.015	2.571	3.163	4.032	4.773
7	1.440	1.943	2.447	2.969	3.707	4.317
8	1.415	1.895	2.365	2.841	3.499	4.029
9	1.397	1.860	2.306	2.752	3.355	3.832
10	1.383	1.833	2.262	2.685	3.250	3.690
20	1.328	1.729	2.093	2.433	2.861	3.174
∞	1.282	1.645	1.960	2.241	2.576	2.807

The radius of the error interval is

$$\Delta x = t \cdot s_{\bar{x}} \; .$$
(7)

We give the results of a repeated measurement in the form

$$x = (\bar{x} \pm \Delta x) \text{ [unit]} , \tag{8}$$

where $\bar{x}$ is the arithmetic mean of the results of measurement repeated N times,
and Δx is the error at a specified level of confidence P (it is usually 0.95):

$$\Delta x = t \cdot s_{\bar{x}} \quad \text{with} \quad s_{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N(N-1)}} \quad \text{and} \quad t \text{ the Student parameter.}$$

Example

We have a pile of resistors with the same nominal resistance which is unknown for us. We take 6 resistors and measure their resistance:

  98 Ω   100 Ω   101 Ω   99 Ω   101 Ω   101 Ω

Calculate the nominal resistance, and the error interval for P = 99% confidence level!

Calculation:

The nominal resistance is calculated as the arithmetic mean:

$$\bar{R} = \frac{98+100+101+99+101+101}{6} = 100 \text{ Ω}.$$

The standard deviation of the mean is calculated using formula (6):

$$s_{\bar{R}} = \sqrt{\frac{(98-100)^2+(100-100)^2+(101-100)^2+(99-100)^2+(101-100)^2+(101-100)^2}{6\cdot 5}} = \sqrt{\frac{8}{30}} \approx 0.5164 \text{ [Ω]};$$

the value of the Student parameter is read from the table:

for N = 6 and P = 0.99 the Student parameter is   t = 4.032;

then the error interval is calculated using formula (7):

$$\Delta R = t \cdot s_{\bar{R}} = 4.032 \cdot 0.5164 = 2.082 \text{ [Ω]}.$$

So the value of the resistances with 99% probability is in the interval

R = (100.0 ± 2.1) Ω .

The error interval is rounded to 2 significant figures, and the mean is given with the same accuracy as the error interval. *If the first figure of the error interval is 1 then 3 significant figures are written.*

2. Propagation of errors

Sometimes a quantity cannot be measured directly, but it depends on other quantities which are easier to measure. *x,y,…* are the measured quantities, and *f* is the calculated quantity. First we have to find the function f(x,y,…). For each measured quantity we know the mean and the error interval with the same *P* confidence level: x = ($\bar{x}$ ± Δx), y = ($\bar{y}$ ± Δy), …, and we want to calculate the mean and the error interval of the quantity *f*. The mean is calculated simply by substituting the mean values into the function f(x,y,..):

$\bar{f}$ = f($\bar{x}$,$\bar{y}$,…).

How is the error interval *Δf* calculated?

We know that a slight change in one of the variables *x,y,…* of a function *f(x,y,…)* causes a change in the function approximately equal to the partial derivative with respect to that variable, multiplied by the change of the variable: $\Delta f_x = \dfrac{\partial f}{\partial x}\Big|_{x_0} \cdot \Delta x$. That is true for the errors, too. The error of the

function *f* caused by the error of one of its variables *x* is $|\Delta f_x| = \left| \dfrac{\partial f}{\partial x}\Big|_{\bar{x}} \cdot \Delta x \right|$.

If there are more variables, the errors caused by all of them can be added, in this case we get **the maximum error**. This maximum error is an overestimation, and it is used only if we want to give our results with maximum confidence. Usually the errors are random, each of them might either increase or decrease the quantity *f*, so the probable error is smaller than the sum of the errors.

We get **the probable error Δf** from the errors of its variables as

$$\Delta f = \sqrt{ \left(\frac{\partial f}{\partial x}\Big|_{\bar{x},\bar{y},…} \cdot \Delta x \right)^2 + \left(\frac{\partial f}{\partial y}\Big|_{\bar{x},\bar{y},…} \cdot \Delta y \right)^2 + \cdots } \ . \tag{9}$$

The partial derivatives have to be calculated at the mean values of the variables, and *Δx, Δy, …* are the errors of the variables *x, y, … .*

Example 1

We have the pile of resistors with nominal resistance

R_1 = (100.0 ± 2.1) Ω

as calculated above and we also have another pile of resistors with nominal resistance

R_2 = (400.0 ± 4.3) Ω   (this error interval is also given for 99% confidence level).

Assume we take one resistor from the first and one from the second pile and connect them in

A. series;

B. parallel.

Calculate the resultant with the error interval for the same confidence level (i.e. calculate the interval in which the resultant will be with 99% probability).

Calculation:

data: $\bar{R_1}$ = 100.0 Ω ; ΔR_1 = 2.1 Ω; $\bar{R_2}$ = 400.0 Ω ; ΔR_2 = 4.3 Ω.

A. Series connection:

The function *f(x,y,...)* how the series resultant R_s is calculated from the resistances R_1 and R_2 is:

$R_s(R_1,R_2) = R_1 + R_2$.

The mean of the series resultant is: $\overline{R}_s = \overline{R}_1 + \overline{R}_2 = 100.0 + 400.0 = 500.0\ \Omega$.

The partial derivative of the function $R_s(R_1,R_2) = R_1 + R_2$ with respect to R_1 and R_2, respectively:

$$\frac{\partial R_s}{\partial R_1} = \frac{\partial R_s}{\partial R_2} = 1.$$

The error interval for the series resultant:

$$\Delta R_s = \sqrt{\left(\frac{\partial R_s}{\partial R_1}\right)^2 \cdot (\Delta R_1)^2 + \left(\frac{\partial R_s}{\partial R_2}\right)^2 \cdot (\Delta R_2)^2} = \sqrt{1^2 \cdot 2.1^2 + 1^2 \cdot 4.3^2} = 4.785\ \Omega.$$

So the series resultant with 99% probability will be in the interval

$R_s = (\ 500.0 \pm 4.8\ )\ \Omega$.

B. Parallel connection:

The function *f(x,y,...)* how the parallel resultant R_p is calculated from the resistances R_1 and R_2 is:

$$R_p(R_1,R_2) = \frac{R_1 \cdot R_2}{R_1 + R_2}\ .$$

The mean of the parallel resultant is:

$$\overline{R}_p = \frac{100.0 \cdot 400.0}{100.0 + 400.0} = 80.0\ \Omega.$$

The partial derivative of the function $R_p(R_1,R_2) = \dfrac{R_1 \cdot R_2}{R_1 + R_2}$ with respect to R_1 and R_2, respectively:

$$\frac{\partial R_p}{\partial R_1} = \frac{R_2 \cdot (R_1 + R_2) - R_1 \cdot R_2 \cdot 1}{(R_1 + R_2)^2} = \left(\frac{R_2}{R_1 + R_2}\right)^2, \quad \text{and similarly} \quad \frac{\partial R_p}{\partial R_2} = \left(\frac{R_1}{R_1 + R_2}\right)^2.$$

By substituting the mean values $\overline{R}_1 = 100.0\ \Omega$ and $\overline{R}_2 = 400.0\ \Omega$ we get

$$\frac{\partial R_p}{\partial R_1} = \left(\frac{400,0}{100,0 + 400,0}\right)^2 = 0.64 \quad \text{and} \quad \frac{\partial R_p}{\partial R_2} = \left(\frac{100,0}{100,0 + 400,0}\right)^2 = 0.04.$$

The error interval for the parallel resultant:

$$\Delta R_p = \sqrt{\left(\frac{\partial R_p}{\partial R_1}\right)^2 \cdot (\Delta R_1)^2 + \left(\frac{\partial R_p}{\partial R_2}\right)^2 \cdot (\Delta R_2)^2} = \sqrt{0.64^2 \cdot 2.1^2 + 0.04^2 \cdot 4.3^2} = 1.355\ \Omega.$$

So, the parallel resultant with 99% probability will be in the interval

$R_p = (\ 80.0 \pm 1.36\ )\ \Omega$.

Example 2

We have measured the electromotive force of a battery six times:
   12.1 V   12.2 V   12.2 V   12.1 V   11.8 V   12.2 V
a) Calculate the value of the emf E with 97.5% confidence level!
b) The battery was loaded with a resistor having a nominal resistance of R = 40.14 Ω with a 97.5% tolerance of ΔR = 0.66 Ω, and the current was measured (with an ideal ammeter) several times, then the current was calculated as
   I = (0.2512 ± 0.0048) A;   P = 97.5%.
Calculate the internal resistance R_i of the battery with 97.5% confidence level!

Calculation

a) $\bar{E}$ = 12.1 V;

$$S_{\bar{E}} = \sqrt{\frac{(12.1-12.1)^2+(12.2-12.1)^2+(12.2-12.1)^2+(12.1-12.1)^2+(11.8-12.1)^2+(12.2-12.1)^2}{6\cdot5}} \approx 0.06325 \text{ V};$$

Student parameter: t(N=6; P=0.975) = 3.163;

$\Delta E = t\cdot S_{\bar{E}}$ = 3.163·0.06325 = 0.2000 V;

E = (12.10 ± 0.20) V.

b)   E = I (R + R_i)   → R_i = E/I − R   is the function R_i(E,R,I).

$\bar{R_i}$ = 12.10/0.2512 − 40.14 = 8.029 Ω is the mean value of the internal resistance.

The partial derivatives:

$$\frac{\partial R_i}{\partial E} = \frac{\partial (E/I-R)}{\partial E} = \frac{1}{I} = \frac{1}{0{,}2512 \text{ A}} = 3.981 \text{ A}^{-1} \qquad\qquad \Delta E = 0.20 \text{ V};$$

$$\frac{\partial R_i}{\partial R} = \frac{\partial (E/I-R)}{\partial R} = -1 \qquad\qquad\qquad\qquad \Delta R = 0.66 \text{ Ω};$$

$$\frac{\partial R_i}{\partial I} = \frac{\partial (E/I-R)}{\partial I} = -\frac{E}{I^2} = -\frac{12{,}10 \text{ V}}{(0{,}2512 \text{ A})^2} = -191.8 \text{ Ω/A} \qquad \Delta I = 0.0048 \text{ A};$$

$$\Delta R_i = \sqrt{\left(\tfrac{\partial R_i}{\partial E}\right)^2 \cdot \Delta E^2 + \left(\tfrac{\partial R_i}{\partial R}\right)^2 \cdot \Delta R^2 + \left(\tfrac{\partial R_i}{\partial I}\right)^2 \cdot \Delta I^2} = \sqrt{3{,}981^2\cdot0{,}20^2+ (-1)^2\cdot0.66^2+ (-191.8)^2\cdot0{,}0048^2}$$

ΔR_i = 1.385 Ω.

The internal resistance with 97.5% confidence level error interval is

R_i = (8.03 ± 1.38) Ω.

Example 3

Six friends measured the length L and the diameter D of a cylindrical pen, and calculated the values of L and D with a 90% confidence level error interval:

L = (162.4 ± 1.5) mm;
D = (11.6 ± 0.13) mm.

Calculate the volume of the pencil with the same confidence level!

Calculation

The volume is calculated as:

$$V = (D/2)^2 \, \pi \, L = D^2 \, \pi \, L \,/\, 4 \,.$$

$$\overline{V} = (11.6 \text{ mm} / 2)^2 \cdot \pi \cdot 162.4 \text{ mm} = 17163 \text{ mm}^3.$$

Partial derivatives:

$$\frac{\partial V}{\partial L} = \frac{\partial (D^2 \pi L \,/\, 4)}{\partial L} = D^2 \, \pi \,/\, 4 = 11.6^2 \, \pi \,/\, 4 = 105.7 \text{ mm}^2; \qquad\qquad \Delta L = 1.5 \text{ mm};$$

$$\frac{\partial V}{\partial D} = \frac{\partial (D^2 \pi L \,/\, 4)}{\partial D} = 2D \, \pi \, L \,/\, 4 = 11.6 \, \pi \cdot 162.4 \,/\, 2 = 2959 \text{ mm}^2; \qquad \Delta D = 0.13 \text{ mm};$$

$$\Delta V = \sqrt{\left(\frac{\partial V}{\partial L} \cdot \Delta L\right)^2 + \left(\frac{\partial V}{\partial D} \cdot \Delta D\right)^2} = \sqrt{(105.7 \cdot 1.5)^2 + (2959 \cdot 0.13)^2} = 416.1 \text{ mm}^3.$$

V = (171.6 ± 4.2)·10³ mm³ = (171.6 ± 4.2) cm³ .

Example 4

We want to know the density of some fluid. We pour it in a beaker and measure its height. This is what we measured:

        3.9 cm   3.6 cm   3.7 cm   3.8 cm   4.0 cm

a) Calculate the height h of the fluid with 80% confidence level!

b) Calculate the density of the fluid with 80% confidence level!

The inner diameter of the beaker is

        d = (5.26 ± 0.12) cm;

the mass of the empty beaker is

        m_b = (82.33 ± 0.10) g;

the mass of the beaker filled with the fluid is

        M = (151.74 ± 0.10) g.

All error intervals belong to P = 0.80.

Calculation

a) $\bar{h} = \dfrac{3.9 + 3.6 + 3.7 + 3.8 + 4.0}{5} = 3.8 \text{ cm}$

$s_{\bar{h}} = \sqrt{\dfrac{(3.9-3.8)^2 + (3.6-3.8)^2 + (3.7-3.8)^2 + (3.8-3.8)^2 + (4.0-3.8)^2}{5\cdot 4}} = \sqrt{\dfrac{0.1}{20}} = 0.07071 \text{ cm}$

Student parameter read from the table: t(N=5; P=0.80) = 1.553;

$\Delta \bar{h} = t \cdot s_{\bar{h}} = 1.553 \cdot 0.07071 = 0.1098$ cm;

h = (3.80 ± 0.11) cm.

b) The mass of the fluid is $M - m_b$; its volume is $(d/2)^2\, \pi \cdot h$,

so the function that gives the density in terms of the measured quantities is:

$\rho(h, d, m_b, M) = (M - m_b) / ((d/2)^2\, \pi \cdot h) = 4\,(M - m_b) / (d^2\, \pi \cdot h).$

$\bar{\rho} = 4 \cdot (151.74 - 82.33) / (5.26^2 \cdot \pi \cdot 3.80) = 0.8406 \text{ g/cm}^3$.

The partial derivatives are:

$\dfrac{\partial \rho}{\partial h} = \dfrac{\partial \left[\{4(M-m_b)/(d^2 \cdot \pi)\} \cdot h^{-1} \right]}{\partial h} = -\dfrac{4(M-m_b)}{d^2 \cdot \pi \cdot h^2} = -\dfrac{\rho}{h} = -0.2212 \text{ g/cm}^4;$ Δh = 0.11 cm;

$\dfrac{\partial \rho}{\partial d} = \dfrac{\partial \left[\{4(M-m_b)/(\pi \cdot h)\} \cdot d^{-2} \right]}{\partial d} = -\dfrac{2 \cdot 4(M-m_b)}{d^3 \cdot \pi \cdot h} = -\dfrac{2\rho}{d} = -0.3196 \text{ g/cm}^4;$ Δd = 0.12 cm;

$\dfrac{\partial \rho}{\partial m_b} = \dfrac{\partial \left[4(M-m_b)/(d^2 \cdot \pi \cdot h) \right]}{\partial m_b} = -\dfrac{4}{d^2 \cdot \pi \cdot h} = -\dfrac{\rho}{M-m_b} = -\dfrac{1}{V} = -0.01211 \text{ cm}^{-3};$ Δm_b = 0.10 g;

$\dfrac{\partial \rho}{\partial M} = \dfrac{\partial \left[4(M-m_b)/(d^2 \cdot \pi \cdot h) \right]}{\partial M} = \dfrac{4}{d^2 \cdot \pi \cdot h} = \dfrac{\rho}{M-m_b} = \dfrac{1}{V} = 0.01211 \text{ cm}^{-3};$ ΔM = 0.10 g;

$\Delta\rho = \sqrt{\left(\dfrac{\partial \rho}{\partial h} \cdot \Delta h \right)^2 + \left(\dfrac{\partial \rho}{\partial d} \cdot \Delta d \right)^2 + \left(\dfrac{\partial \rho}{\partial m_b} \cdot \Delta m_b \right)^2 + \left(\dfrac{\partial \rho}{\partial M} \cdot \Delta M \right)^2} = 0,04543 \text{ g/cm}^3;$

ρ = (0.841 ± 0.045) g/cm³.