Laser Physics 17.

Transformation of Gaussian beam
Continuous laser operation

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Transformation of Gaussian beams

$\underline{TEM_{00}\text{ mode}}$ – transmission through arbitrary optical system (*ABCD*)

In the paraxial approximation the optical system is completely characterized by the 2 x 2 ray-transfer matrix. The Gaussian-beam with *q* complex parameter is a special spherical wave. Examine firstly the transmission of a spherical wave through an arbitrary optical system!

r_1 is the distance and r_1' is the angle of the ray to the optical axis, because of the paraxial approximation

$$\sin r_1' \cong tg\ r_1' \cong r_1',$$

$$r_1' > 0$$

$$R_1 = \frac{r_1}{r_1'}, \quad R_2 = \frac{r_2}{r_2'}$$

If angle to the +*z* direction, *R* is positive, if the center is left to the wavefront

Spherical wave from P_1 to P_2 through the optical system

$$r_2 = Ar_1 + Br_1', \quad r_2' = Cr_1 + Dr_1' \quad \rightarrow \quad \frac{r_2}{r_2'} = \boxed{R_2 = \frac{AR_1 + B}{CR_1 + D}}$$

transmission of a spherical wave

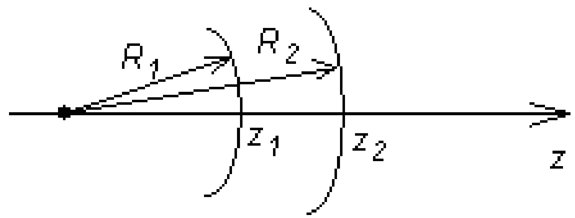
Transformation of Gaussian beams

TEM$_{00}$ mode– transmission through arbitrary optical system (*ABCD*)

Transmission of spherical wave:

$$R_2 = \frac{AR_1 + B}{CR_1 + D}$$

1) Free-space propagation (n = 1)



$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 & \frac{L}{n} \\ 0 & 1 \end{vmatrix}, \quad R_2 = R_1 + (z_2 - z_1)$$

2) Transmission through thin lens

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix} \quad R_2 = \frac{R_1}{-\dfrac{R_1}{f} + 1} \quad \rightarrow \quad \frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

Transformation of Gaussian beams

TEM$_{00}$ mode– transmission through arbitrary optical system (*ABCD*)

Transformation of a Gaussian beam with *q* parameter:

$$\boxed{q_2 = \frac{Aq_1 + B}{Cq_1 + D}}$$

1) Free-space propagation (*n* = 1)

$$q_2 = q_1 + (z_2 - z_1)$$

2) Transmission through thin lens

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \qquad\qquad \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)},$$

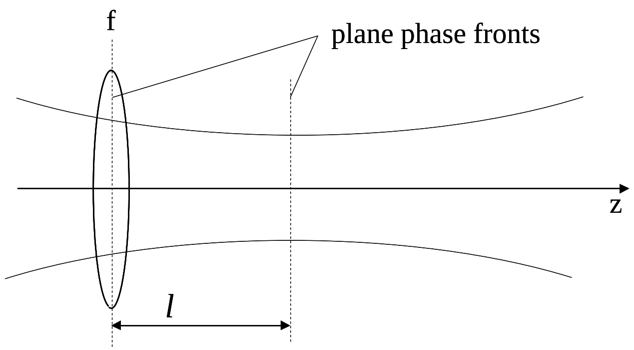
The wavefront is plane at the beam waist therefore *q* is there purely imaginary. The place of the transformed beam waist can be determined from the condition:

real part of $q^{-1} = 0$.

Transformation of Gaussian beams

2) Transmission through thin lens (cont.)

We put a thin lens into the beam waist of a Gaussian beam. Calculate the place of the new beam waist after the lens.



On the lens the phase front of the beam is plane.

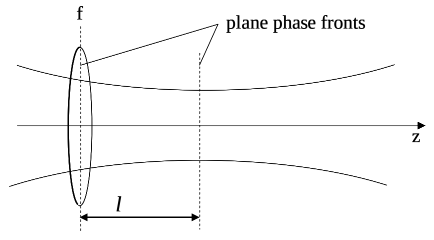
$$R_1 = \infty, \quad \frac{1}{q_1} = -i\frac{\lambda}{\pi w_0^2}, \quad q_1 = i\frac{\pi w_0^2}{\lambda} = iz_0$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 & l \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix} = \begin{vmatrix} 1-\frac{l}{f} & l \\ -\frac{1}{f} & 1 \end{vmatrix}$$

We are looking for the distance *l*, where the q_2 parameter is full complex, i.e. $R_2=\infty$ (the place of the new beam waist).

Transformation of Gaussian beams

2) Transmission through thin lens (cont.)



$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1-\frac{l}{f} & l \\ -\frac{1}{f} & 1 \end{vmatrix} \qquad q_2 = \frac{Aq_1+B}{Cq_1+D}$$

$$\frac{1}{q_2} = \frac{1-\frac{q_1}{f}}{\left(1-\frac{l}{f}\right)q_1+l} = \frac{1-\frac{iz_0}{f}}{iz_0\left(1-\frac{l}{f}\right)+l} = \frac{\left(1-\frac{iz_0}{f}\right)\left\{-iz_0\left(1-\frac{l}{f}\right)+l\right\}}{\left[iz_0\left(1-\frac{l}{f}\right)+l\right]\left[-iz_0\left(1-\frac{l}{f}\right)+l\right]}$$

$$l-\frac{z_0^2}{f}\left(1-\frac{l}{f}\right)=0$$

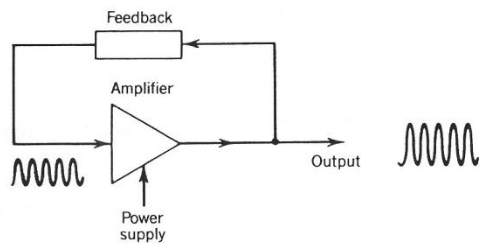
The real part of q_2 has to be zero!

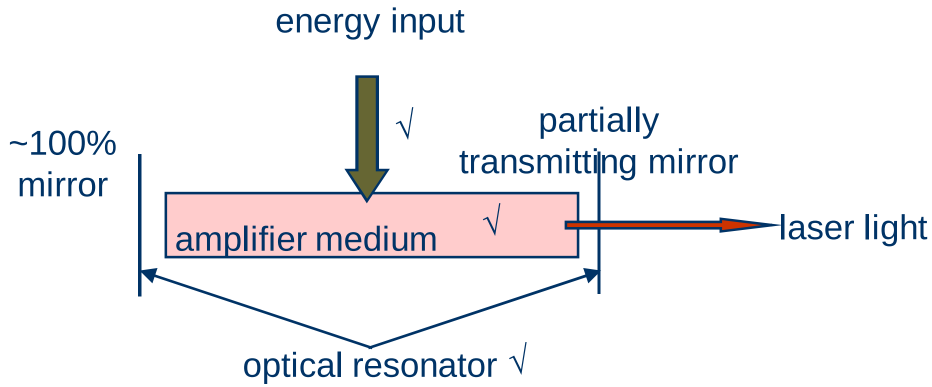
$$l\left(1+\frac{z_0{}^2}{f^2}\right)=\frac{z_0{}^2}{f} \quad\rightarrow\quad l=\frac{\frac{z_0{}^2}{f}}{1+\frac{z_0{}^2}{f^2}} \qquad \boxed{l=\frac{f}{1+\left(\frac{f}{z_0}\right)^2}}$$

The new beam waist is not in the focal plane. Only when the depth of focus of the beam $2z_0$ >> f, behaves the beam as a plane wave.

Continuous laser operation

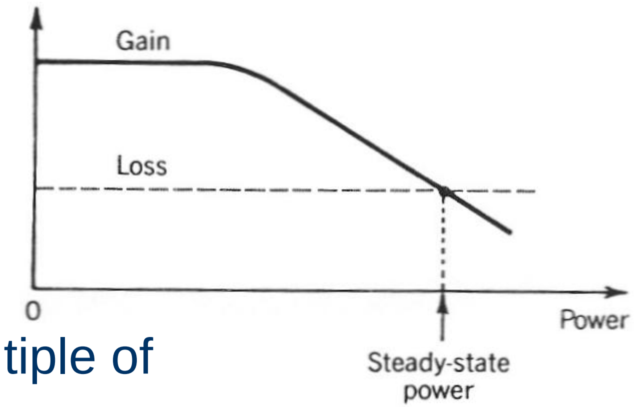
<u>What is needed to make a laser?</u>





The laser is an optical oscillator. It comprises a resonant optical amplifier whose output is fed back with matching phase. Laser operation builds up gradually: spontaneous emission → stimulated emission → feedback → increasing photon flux density → saturation → steady state operation. Two conditions must be satisfied:

- Amplifier gain > loss in the feedback system (<u>gain condition</u>),
- Feedback input phase matches the phase of the original input, the total phase shift in a single round trip is a multiple of 2π (<u>phase condition</u>).



Continuous laser operation

Coherent optical amplifier – reminder

Steady-state population inversion in the absence of amplifier radiation

small signal gain:

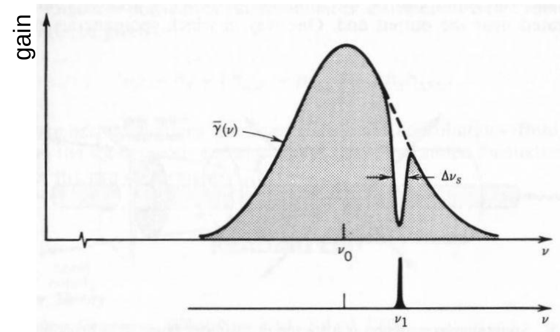
$$\gamma_0(\nu) = N_0 \sigma(\nu) = N_0 \frac{\lambda^2}{8\pi t_{sp}} g(\nu),$$

homogeneous amplifier:

$$\gamma(\nu) = \frac{\gamma_0(\nu)}{1 + \Phi/\Phi_s(\nu)}, \qquad \frac{1}{\Phi_s(\nu)} = \tau_s \sigma(\nu)$$

saturation photon-flux density

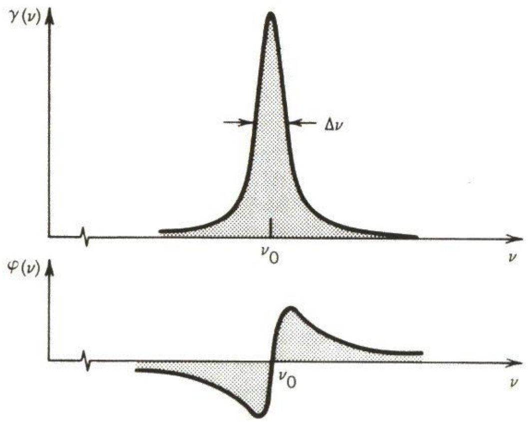
saturated inhomogeneous amplifier:

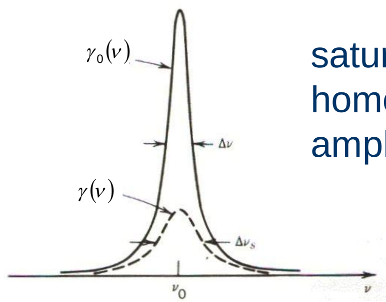


gain and

phase shift

$$\varphi(\nu) = \frac{\nu - \nu_0}{\Delta\nu} \gamma(\nu).$$





saturated homogeneous amplifier

Continuous laser operation

Conditions of the laser operation – gain condition

Gain of the medium > loss of the feedback system

threshold population difference

$$\boxed{\gamma_0(\nu) > \alpha_r},$$

$$N_0 = \frac{\gamma_0(\nu)}{\sigma(\nu)} > \frac{\alpha_r}{\sigma(\nu)} = N_t$$

$$N_0 > N_t$$

$$N_t = \frac{1}{c\tau_r\sigma(\nu)} \quad \text{while} \quad \frac{1}{\alpha_r c} = \tau_r$$

The steady-state population inversion density without amplifier radiation must be greater than the threshold population difference that is dependent on the loss of the feedback system and the transition cross section!

$$N_t = \frac{8\pi}{\lambda^2 c} \cdot \frac{t_{sp}}{\tau_r} \cdot \frac{1}{g(\nu)}.$$ ν and λ dependence!

Continuous laser operation

<u>Conditions of the laser operation</u> – phase condition (homogeneous medium)

Total phase shift (in the optical resonator and the coherent medium, both have length of L) for a round-trip is $2\pi\, q$, q is integer:

$$2kL + 2L\varphi(\nu) = 2\pi q, \quad \left(k = 2\pi \frac{\nu}{c}, \quad \varphi(\nu) = \frac{\nu - \nu_0}{\Delta\nu}\gamma(\nu)\right) \quad / : \frac{2\pi \cdot 2L}{c}$$

$$\nu + \frac{(\nu - \nu_0)c}{2\pi\Delta\nu}\gamma(\nu) = \underbrace{\frac{c}{2L} \cdot q = \nu_q}_{\text{frequency of cold-resonator mode}},$$

$$\underbrace{\nu + \frac{c}{2\pi}\frac{\nu - \nu_0}{\Delta\nu}\gamma(\nu)}_{\psi(\nu)} = \nu_q.$$

$$\psi(\nu) = \nu_q$$

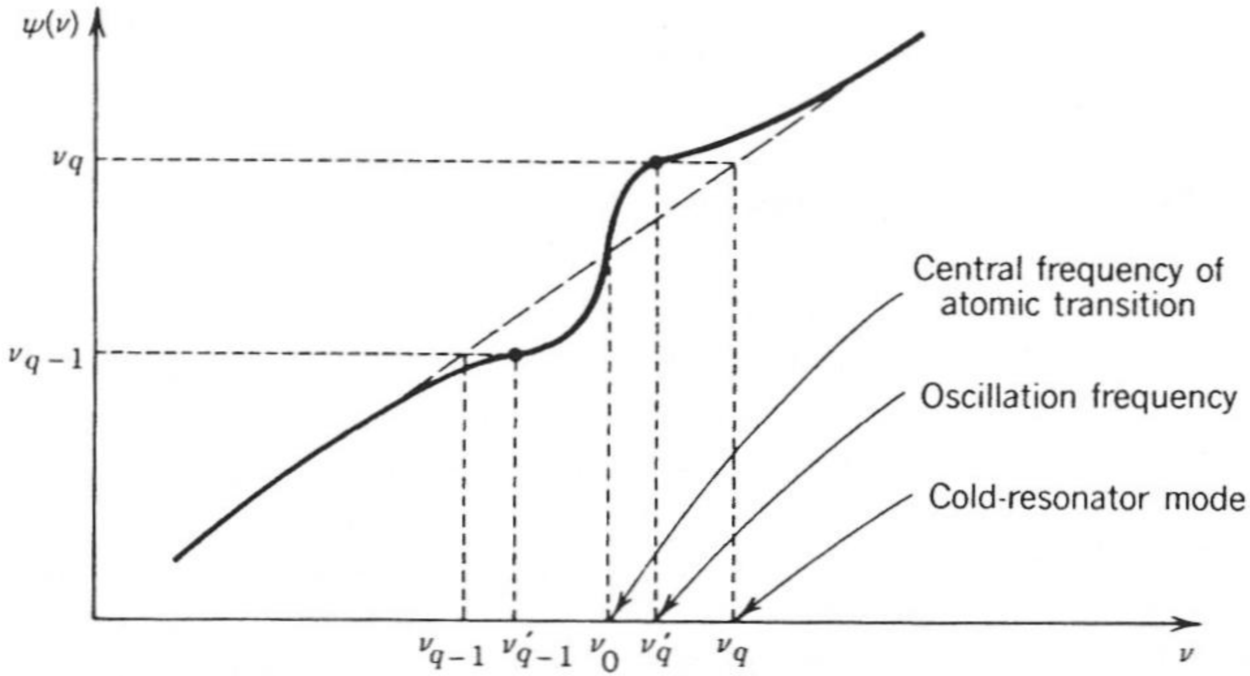
ν_q' is the frequency of the laser mode

Graphical or approximate analytical solutions.

Continuous laser operation

<u>Conditions of the laser operation</u> – phase condition (homogeneous medium, cont.)

Graphical solution



The frequency of laser modes are pulled slightly toward the atomic resonance central frequency!

Continuous laser operation

Conditions of the laser operation – phase condition (homogeneous medium, cont.)

Value of the mode-pulling - on the basis of approximate analytical solution:

$$\nu + \frac{c}{2\pi}\frac{(\nu-\nu_0)}{\Delta\nu}\gamma(\nu) = \nu_q$$

rearrange in the form:

$$\nu = \nu_q - \frac{c}{2\pi}\frac{(\nu-\nu_0)}{\Delta\nu}\gamma(\nu)$$

approximation (right side): $\nu = \nu_q^{'} \approx \nu_q$

$$\nu_q^{'} \approx \nu_q - \frac{c}{2\pi}\frac{(\nu_q-\nu_0)}{\Delta\nu}\gamma(\nu_q)$$

at steady-state operation

$$\gamma(\nu_q) = \alpha_r, \quad \alpha_r = \frac{2\pi}{c}\Delta\nu_r$$

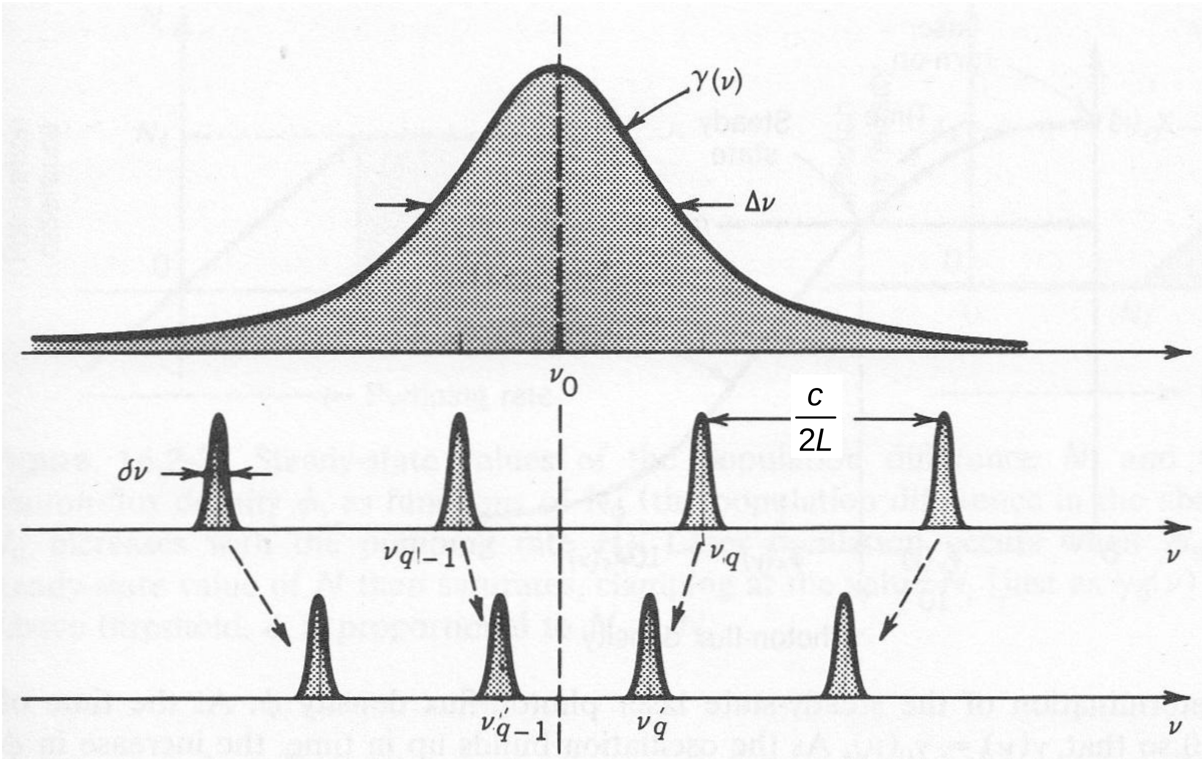
$$\boxed{\nu_q^{'} \approx \nu_q - (\nu_q - \nu_0)\frac{\Delta\nu_r}{\Delta\nu}}$$

Mode-pulling by a fraction of $\Delta\nu_r / \Delta\nu$ of the frequency difference $(\nu_q - \nu_0)$!

The value of the mode-pulling is higher for modes far from the resonance frequency. The central frequency is stable!

Continuous laser operation

Conditions of laser operation – phase condition (homogeneous medium, cont.)



Amplifier gain coefficient

Cold-resonator modes

Laser oscillation modes

Mode-pulling is higher for modes far from the resonance frequency. The central frequency is stable!

Continuous laser operation

Steady-state operation

If $N_0 > N_t$, the laser oscillation may begin. As the photon-flux density increases, the gain coefficient begins to decrease until it reaches the value of the loss coefficient. The gain is clamped to the value of the loss, the inversion density is equal to its threshold value.

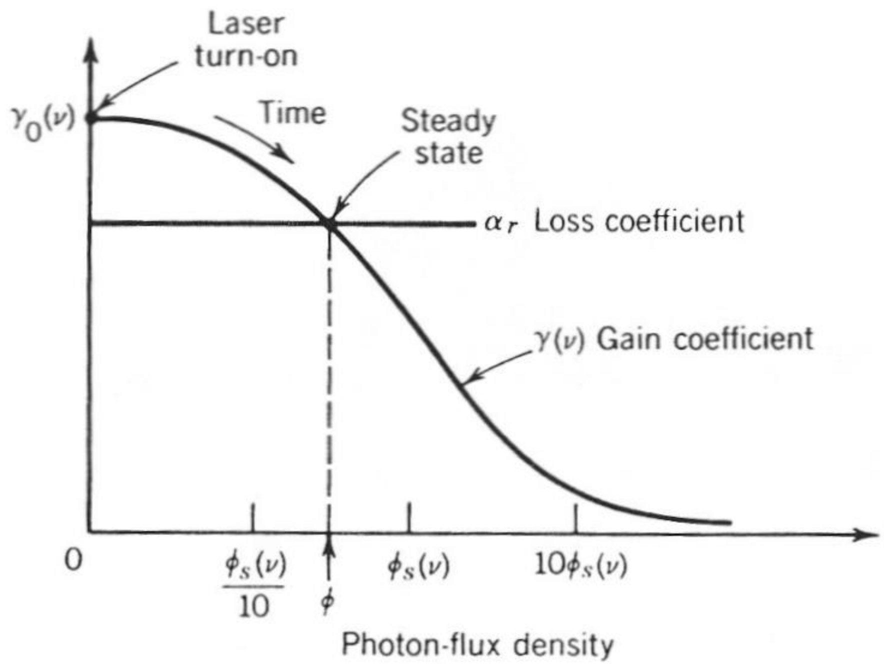
In steady state:

$$\boxed{N = N_t}$$

In homogeneous medium:

$$\gamma(\nu) = \alpha_r = \frac{\gamma_0(\nu)}{1 + \Phi / \Phi_s(\nu)},$$

$$\alpha_r [1 + \Phi / \Phi_s] = \gamma_0(\nu)$$



Continuous laser operation

Steady-state operation – homogeneous medium (cont.)

$$\alpha_r = \frac{\gamma_0(\nu)}{1 + \Phi / \Phi_s(\nu)},$$

$$\alpha_r [1 + \Phi / \Phi_s] = \gamma_0(\nu)$$

$$\Phi = \Phi_s(\nu)\left(\frac{\gamma_0(\nu)}{\alpha_r} - 1\right), \quad \text{ha} \quad \gamma_0(\nu) > \alpha_r,$$ Parameter: gain coefficient

$$\Phi = 0, \quad \text{ha} \quad \gamma_0(\nu) < \alpha_r,$$

$$\Phi = \Phi_s(\nu)\left(\frac{N_0}{N_t} - 1\right), \quad \text{ha} \quad N_0 > N_t,$$ Parameter: inversion density

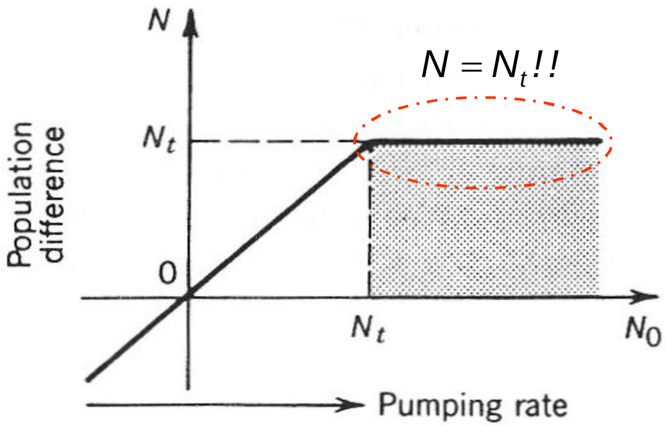
$$\Phi = 0, \quad \text{ha} \quad N_0 < N_t.$$

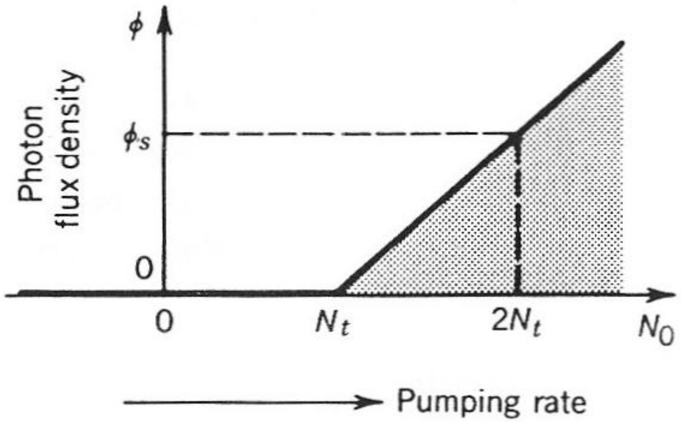
Φ gives the mean number of photons per second crossing through a unit area in both directions in the laser (random fluctuations are possible). We neglect here the effect of the spontaneous emission. Photon-flux density traveling in a given direction is $\Phi / 2$!

Continuous laser operation

<u>Steady-state operation</u> – homogeneous medium (cont.)

Below threshold Φ is zero, as the pumping rate N_0 increases the amount of spontaneously emitted photons increases, but there is no laser oscillation!





$$\Phi_s(\nu)\left(\frac{N_0}{N_t}-1\right)$$

Above threshold Φ is directly proportional to the initial population difference N_0 and increases with the pumping rate (remember: $N_0 = N_2 - N_1 = R_2\tau_2\left(1-\frac{\tau_1}{\tau_{21}}\right)+R_1\tau_1.$).
If N_0 is twice the threshold value N_t, Φ is equal to the saturation value of Φ_s.

Continuous laser operation

Output laser power – homogeneous medium

If the transmittance of the mirror is T, the output photon-flux density, the output intensity, and the output power are:

$$\Phi_{out} = T\frac{\Phi}{2}, \quad I_{out} = h\nu\frac{T\Phi}{2}, \quad P_{out} = I_{out}A,$$

where A is the cross-sectional area of the laser beam.

Optimization of the output photon-flux density (or the output power)

The useful (output) photon-flux density causes loss in the internal photon-flux density, therefore contributes to the losses $\rightarrow$ the threshold inversion density increases. Without outcoupling (T = 0%) the inner photon-flux density would be higher. The increase of the mirror transmittance causes first an increase in the output photon-flux density, then the increase of the loss will decrease the steady-state photon-flux density. There is an optimal transmittance that maximizes the laser output intensity.

Continuous laser operation

Optimization of the output photon-flux density – homogeneous medium (cont.)

$$\Phi_{out} = \frac{1}{2}\Phi_s(\nu)T\left(\frac{\gamma_0(\nu)}{\alpha_r} - 1\right).$$

Depends on T, we write in explicit form!

$$\alpha_r = \frac{-\ln R_1 R_2}{2L}$$

$$\alpha_{m1} = -\frac{1}{2L}\ln R_1 = -\frac{1}{2L}(1-T)$$

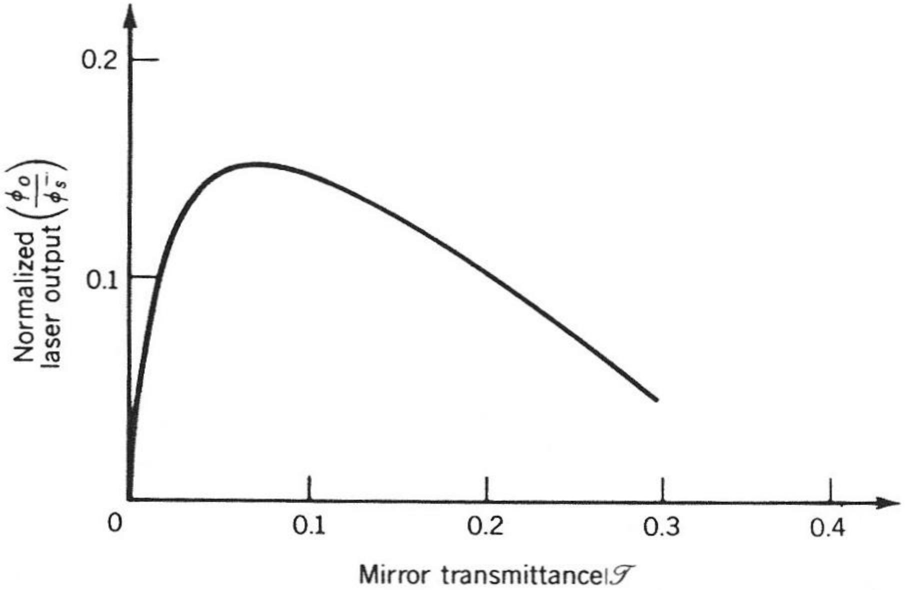
$$\alpha_r = \alpha_s + \alpha_{m2} - \frac{1}{2L}(1-T), \quad \alpha_{m2} = -\frac{1}{2L}\ln R_2.$$

$$\Phi_{out} = \frac{1}{2}\Phi_s(\nu)T\left(\frac{2\gamma_0(\nu)L}{C - \ln(1-T)} - 1\right), \quad C = 2(\alpha_s + \alpha_{m2})L.$$

$$\frac{\partial \Phi}{\partial T} = 0 \quad \rightarrow \quad T_{opt} \text{ can be determined.} \quad T << 1, \quad \text{therefore} \quad \ln(1-T) \approx -T$$

Continuous laser operation

Optimization of the output photon-flux density – homogeneous medium (cont.)



Parameters: $2\gamma_0(\nu)L = 0.5$ $\quad T_{opt} \approx 0.08$

$$C = 2(\alpha_s + \alpha_{m2})L = 0.02$$

Continuous laser operation

Spectral distribution

It is determined by the atomic lineshape of the active medium and by the resonator modes.

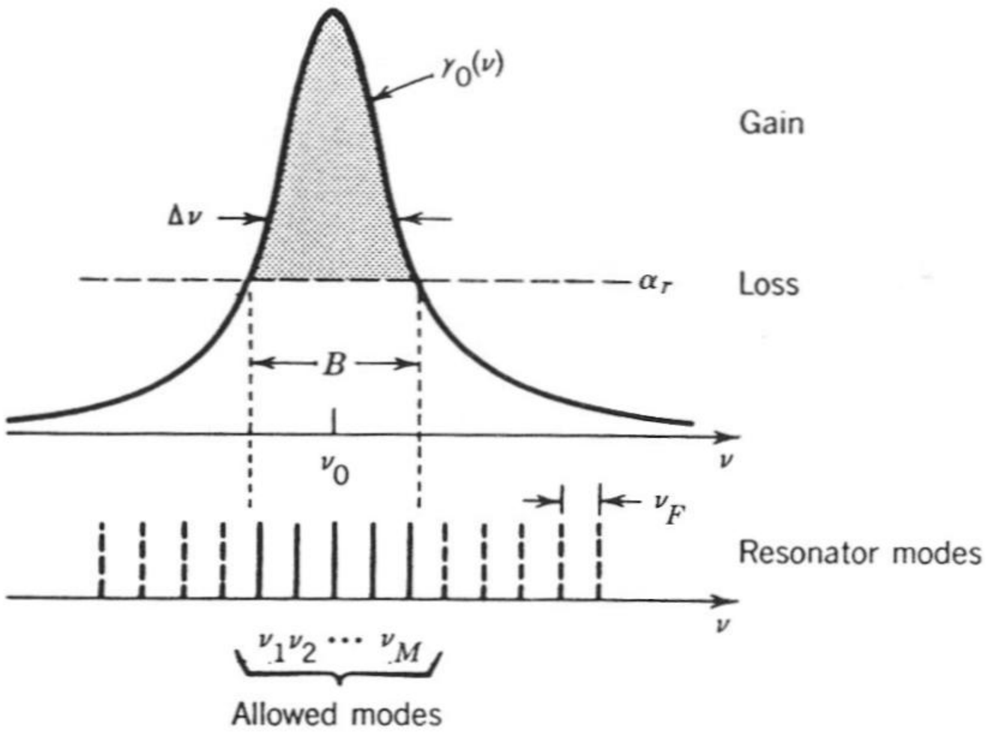
On the basis of the gain condition the operational bandwidth B can be defined, within B there are oscillating laser modes.

B depends on $\Delta\nu$ and on the ratio of $\gamma_0(\nu)$ / α_r.

ν_1, ν_2, … ν_M are laser modes, the number of modes is M:

$$M \approx \frac{B}{\nu_F}, \quad \nu_F = \frac{c}{2L}$$

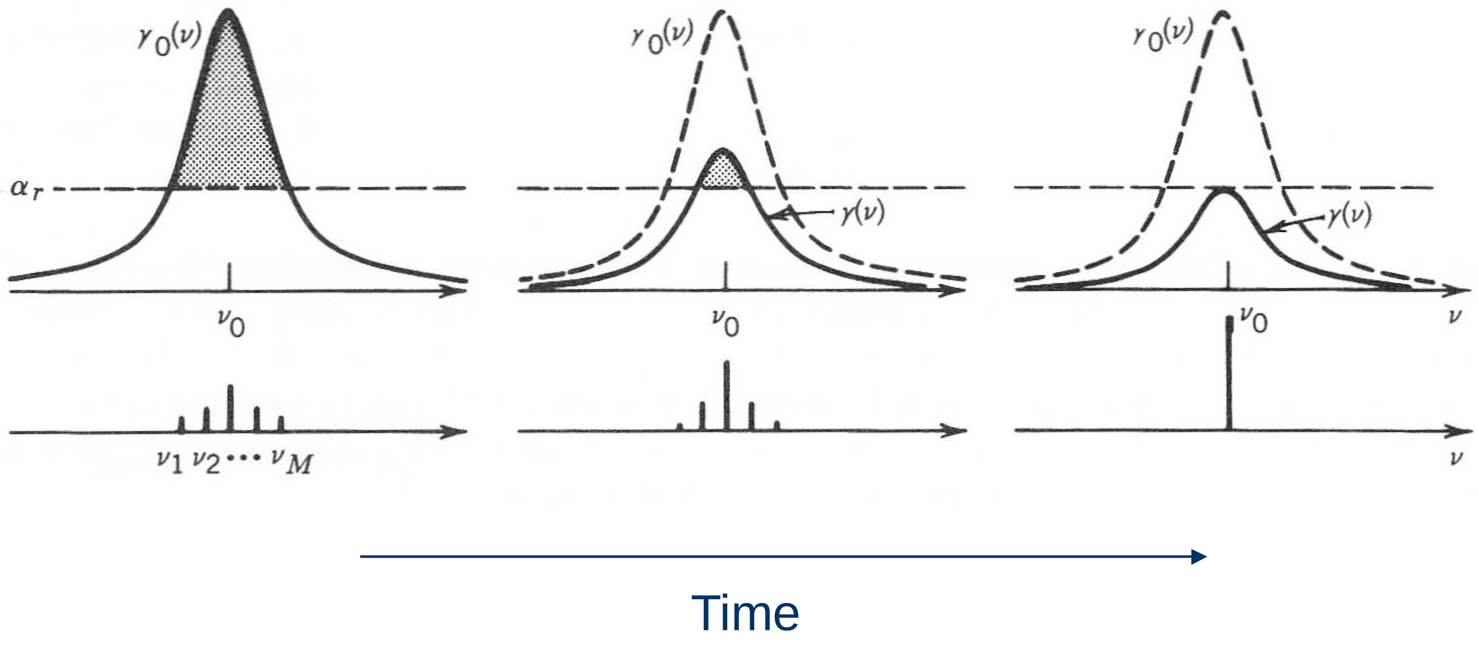
The number of modes depends on the type of broadening!



Continuous laser operation

Spectral distribution – homogeneously broadened medium

The interaction with light is the same for all particle! All particles interact with light of any frequency within *B*. Single mode oscillation?



Continuous laser operation

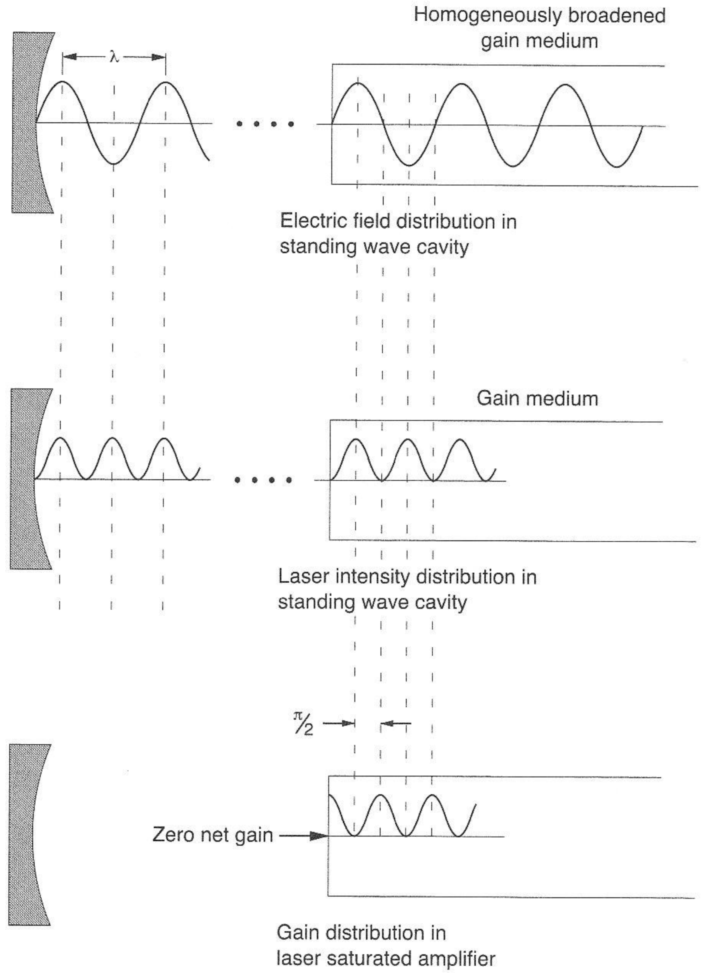
Spectral bandwidth – homogeneously broadened medium (cont.)

In practice the spatial distribution of the mode amplitude has to be taken into account.

In the standing wave field the gain saturates where the mode amplitude is high. Where the amplitude is zero there is no saturation and another modes can start to oscillate.

In practice the multi-mode operation is typical also in homogeneous medium. This phenomenon is called „spatial hole burning".

Solid state lasers are typically homogeneous amplifiers!

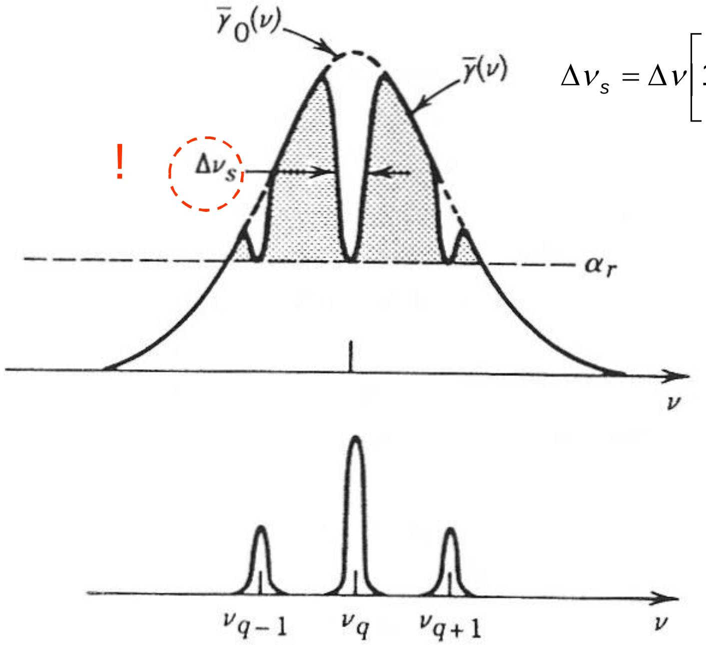


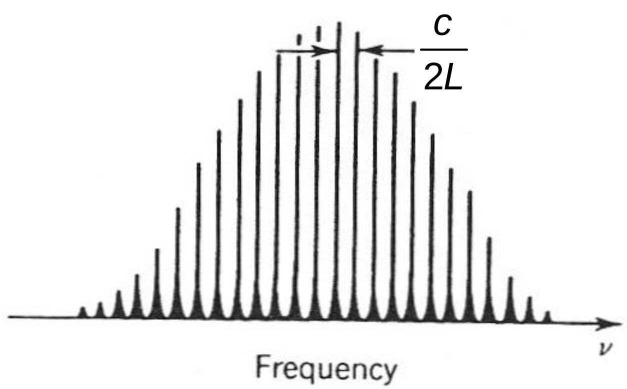
Continuous laser operation

Spectral bandwidth – inhomogeneous amplifier

"Spectral hole burning" because of the independently oscillating modes. The number of modes is typically larger than in homogeneously broadened media.

$$\Delta\nu_s = \Delta\nu\left[1+\frac{\Phi}{\Phi_s(\nu_0)}\right]^{1/2}$$





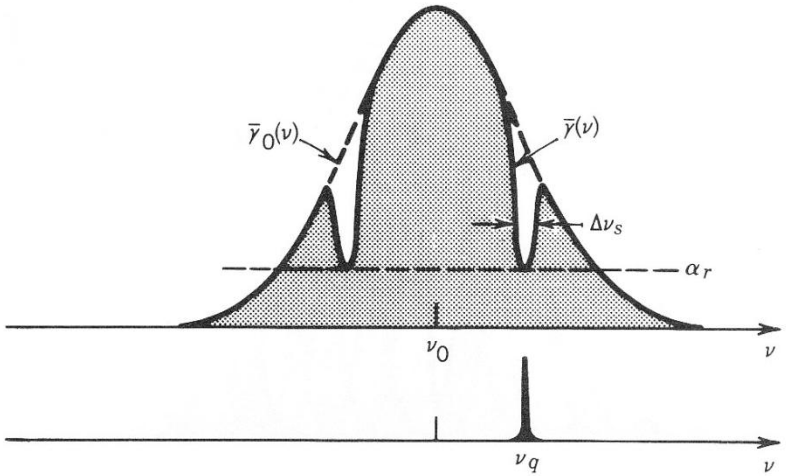
In practice multi-mode operation is general in lasers.

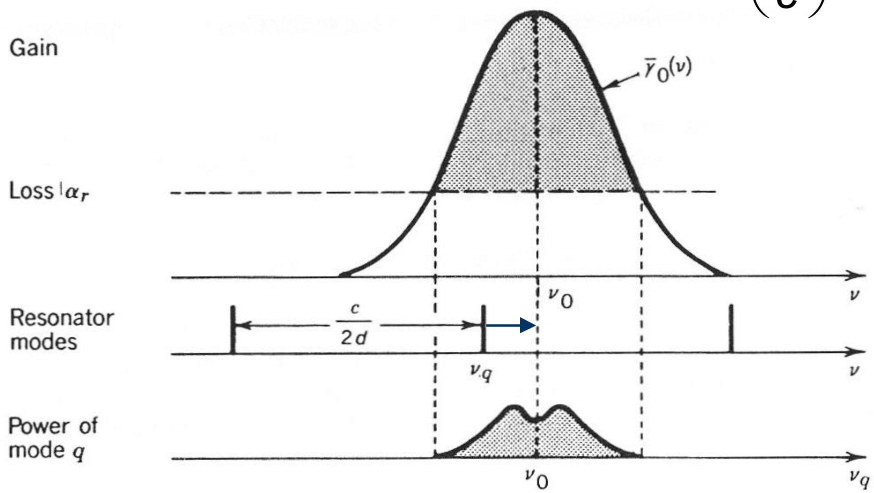
Continuous laser operation

Spectral hole burning in a Doppler-broadened medium

The lineshape of a gas at temperature T arises from the collection of Doppler-shifted emissions of the moving particles.

$$\nu_0 \rightarrow \nu_0 \pm \left(\frac{v}{c}\right)\nu_0,$$





Saturation can be achieved by high intensity ν_q frequency field at two frequencies symmetrically. The mode travels back and forth, the gain of two group of atoms saturates, there will be two holes!

At resonance frequency only one hole burns $\rightarrow$ the power of the mode decreases (Lamb-dip).