Laser Physics 16.

Stability of resonators
Properties of Gaussian beams

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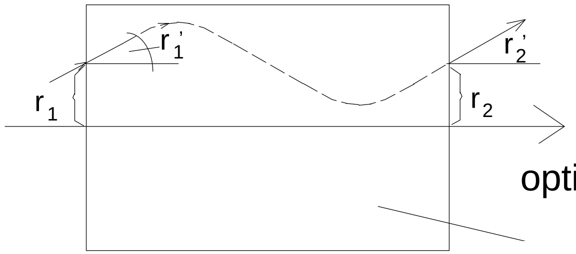
Stability of resonators

Ray tracing in paraxial approximation

The approximation is valid for rays that remain close to the optical axis and have small divergence angles – resonators with two small size mirrors in a large distance can be considered to be a paraxial system. We have to determine the transfer matrixes



r_1 is the distance of a ray and r_1' is its angle to the optical axes, the following conditions must be fulfilled

$$\sin r_1' \cong tg\, r_1' \cong r_1', \quad r_1' > 0$$

positive if it is the angle to the $+z$ direction.

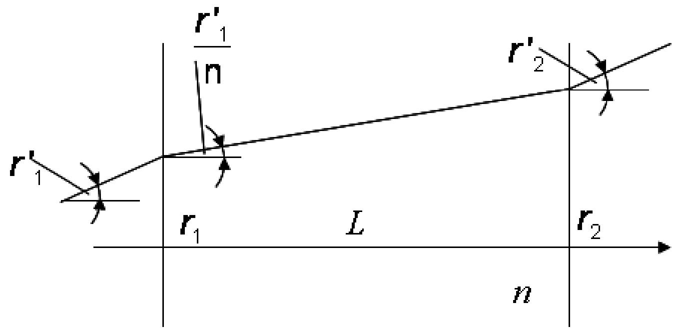
The ray goes through the optical system by linear transformations :

$$\begin{vmatrix} r_2 \\ r_2' \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} r_1 \\ r_1' \end{vmatrix}$$

Transfer matrix of the optical system

Stability of resonators

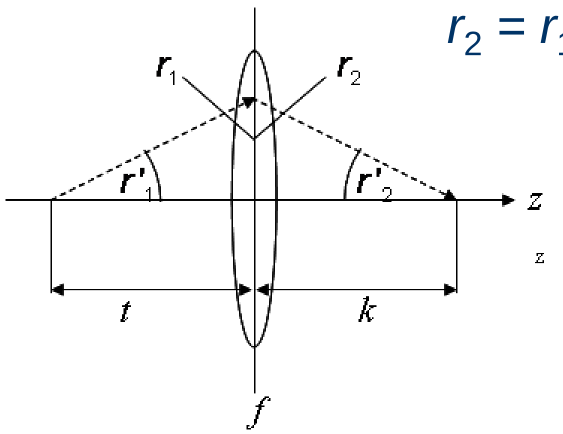
1. Free–space propagation along L of a material with refractive index n



$$r_2 = r_1 + \frac{r_1'}{n}L \qquad r_2' = r_1'$$

$$\begin{vmatrix} 1 & \frac{L}{n} \\ 0 & 1 \end{vmatrix} \qquad D = AD - BC = 1$$

2. Ray propagation through a lens



$r_2 = r_1$ (thin lens)

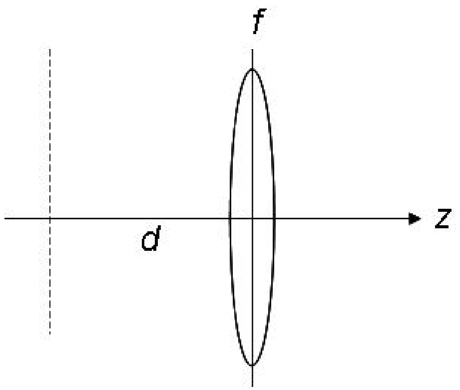
$$t = \frac{r_1}{r_1'}, \quad k = -\frac{r_2}{r_2'}, \quad \frac{1}{t} + \frac{1}{k} = \frac{1}{f}$$

$$\frac{r_1'}{r_1} - \frac{r_2'}{r_2} = \frac{1}{f} \qquad r_1 = r_2, \quad r_2' = -\frac{r_1}{f} + r_1'$$

$$\begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix} \qquad D = 1$$

Stability of resonators

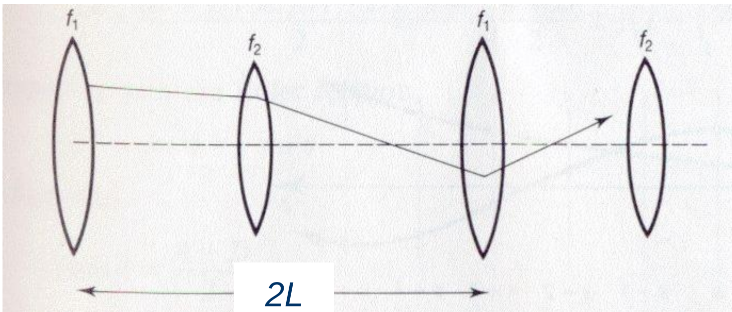
3. Ray propagation through free space *d* and lens *f* (*n* = 1)



$$\begin{vmatrix} r_2 \\ r_2' \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix} \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} \begin{vmatrix} r_1 \\ r_1' \end{vmatrix} = \begin{vmatrix} 1 & d \\ -\frac{1}{f} & -\frac{d}{f}+1 \end{vmatrix} \begin{vmatrix} r_1 \\ r_1' \end{vmatrix} \qquad D = -\frac{d}{f} + 1 + \frac{d}{f} = 1$$

4. Resonator

Instead of the mirrors equivalent lenses can be used → ray propagation though a sequence of lenses:



unit cell

$$R_{1,2} \rightarrow f_{1,2} = \frac{R_{1,2}}{2}$$

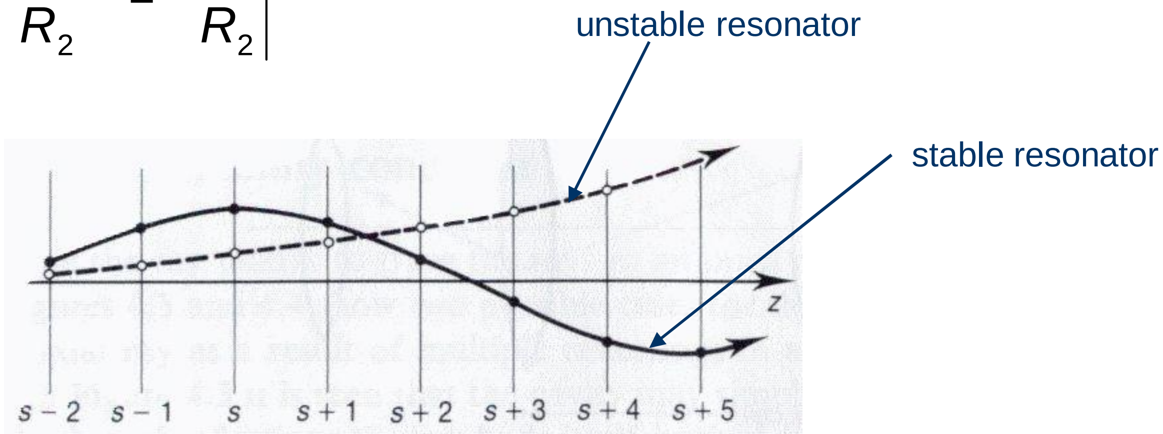
Stability of resonators

For one round-trip (propagation through length L and lens f_1, then length L and lens f_2 – through the unit cell)

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 & L \\ -\dfrac{2}{R_1} & 1-\dfrac{2L}{R_1} \end{vmatrix} \begin{vmatrix} 1 & L \\ -\dfrac{2}{R_2} & 1-\dfrac{2L}{R_2} \end{vmatrix}$$

after n cell

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}^n$$



In the stable cavity the ray returns to the starting position after wandering, the path can be considered sin or cos function. In the unstable cavity there is no return to the starting position, this can be written by an exponential function.

Stability of resonators

Deduction: only for information!!

$$r_{s+1} = A r_s + B r_s' \quad or \quad r_s' = \frac{1}{B}\left(r_{s+1} - A r_s\right)$$

$$r_{s+1}' = C r_s + D r_s' = \frac{1}{B}\left(r_{s+2} - A r_{s+1}\right)$$

$$\frac{1}{B}\left(r_{s+2} - A r_{s+1}\right) = C r_s + \frac{D}{B}\left(r_{s+1} - A r_s\right)$$

$$r_{s+2} - A r_{s+1} = BC r_s + D r_{s+1} - AD r_s, \quad \text{but} \quad AD - BC = 1$$

$$\boxed{r_{s+2} - (A + D) r_{s+1} + r_s = 0}$$

We are looking for the solution of this quadratic recursive equation that is valid for stable and unstable resonators. Stable – periodic function, unstable – exponential function. Suitable solution: $r_s = r_0 e^{i\theta s}$

Stability of resonators

Deduction: only for information!!

Trial function: $r_s = r_0 e^{i\theta s}$, r_0 is the initial coordinate of the ray, θ is a geometric constant of the cavity which has to be real for a stable cavity.

$$r_{s+2} - (A+D) r_{s+1} + r_s = 0$$

$$r_0 \left(e^{i\theta}\right)^{s+2} - r_0 (A+D)\left(e^{i\theta}\right)^{s+1} + r_0 \left(e^{i\theta}\right)^{s} = 0$$

$$r_0 \left(e^{i\theta}\right)^{s} \left[\left(e^{i2\theta}\right) - (A+D) e^{i\theta} + 1\right] = 0$$

In general case r_0 and $e^{i\theta}$ are not zero:

$$\left(e^{i2\theta}\right) - (A+D) e^{i\theta} + 1 = 0$$

quadratic equation

$$e^{i\theta} = \frac{(A+D) \pm \sqrt{(A+D)^2 - 4}}{2} = \frac{A+D}{2} \pm i \left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}$$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

comparing with the Euler relation!

Stability of resonators

Deduction: only for information!!

$$\boxed{\cos\theta = \frac{A+D}{2}}$$

The geometric factor depends only on the *ABCD* matrix of the cavity.

The complete solution for r_s must be real since it is an observable quantity, instead of the trial solution the complete solution is:

$$r_s = r_0 e^{i\alpha} e^{i\theta s} + r_0 e^{-i\alpha} e^{-i\theta s},$$

Including the complex conjugate, where α is an angular constant determining the starting position of the ray. When θ is real then the solution will be oscillatory:

$$r_s = r_{max} \cos(s\theta + \alpha)$$

r_{max} is the maximum displacement of the ray in the cavity.

Stability of resonators

Deduction: only for information!!

$$\boxed{\cos\theta = \frac{A+D}{2}}$$

The condition for stability: $-1 \le \dfrac{A+D}{2} \le +1, \quad 0 \le \dfrac{A+D+2}{4} \le 1$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1 & L \\ -\dfrac{2}{R_1} & 1-\dfrac{2L}{R_1} \end{vmatrix} \begin{vmatrix} 1 & L \\ -\dfrac{2}{R_2} & 1-\dfrac{2L}{R_2} \end{vmatrix}$$

$$A = 1-\frac{2L}{R_2}, \quad D = -\frac{2L}{R_1} + \left(1-\frac{2L}{R_1}\right)\left(1-\frac{2L}{R_2}\right) = 1-\frac{2L}{R_1}-\frac{2L}{R_2}-\frac{2L}{R_1}+\frac{4L^2}{R_1R_2}$$

$$\frac{A+D+2}{4} = 1-\frac{L}{R_1}-\frac{L}{R_2}+\frac{L^2}{R_1R_2} = \left(1-\frac{L}{R_1}\right)\left(1-\frac{L}{R_2}\right)$$

$$g_{1,2} = 1-\frac{L}{R_{1,2}}$$

cavity *g* parameter

The stability condition: $\boxed{0 \le g_1 g_2 \le 1}$

Stability of resonators

The boundaries of stability: axis g_1, g_2 and hyperbola g_1g_2=1. Symmetrical resonators are on the line 45°.

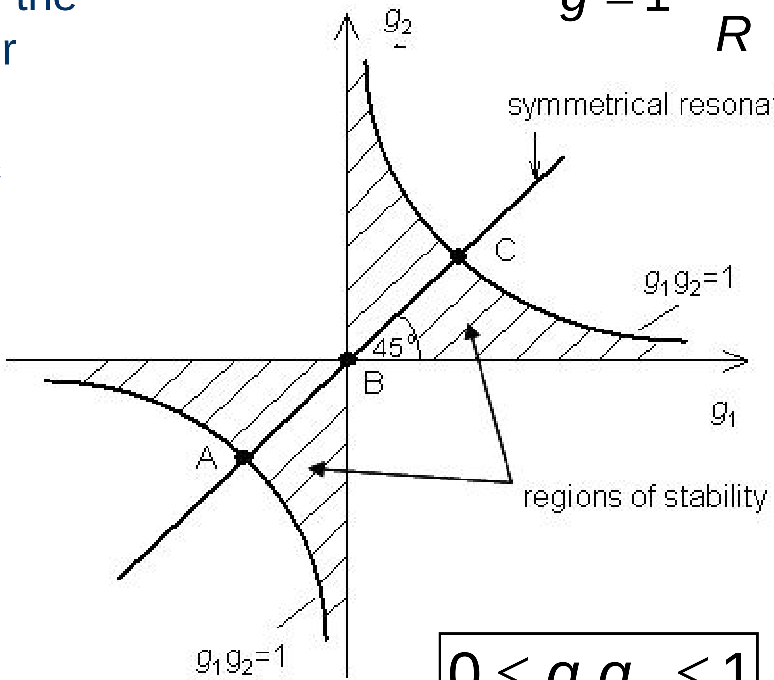
Special geometries at the boundaries of the stability (slight errors in alignment render them unstable):

A) $g_1 = g_2 = -1 \rightarrow L = 2R$, **concentric resonator**, drawback: small spot size in the middle of the cavity

B) $g_1 = g_2 = 0 \rightarrow L = R$, **confocal resonator**, spot size also small

C) $g_1 = g_2 = 1 \rightarrow R = \infty$, **planar resonator**, good space padding

$$g = 1 - \frac{L}{R}$$



$$\boxed{0 \le g_1 g_2 \le 1}$$

Practical resonators: two mirrors with high radius of curvature ((2-10) x *L*) or one plane and one curved mirror of high radius of curvature.

Properties of Gaussian beams

The modes of lossless (*N >>1)* confocal resonator are the Gaussian functions. (rev.)

Normalized field distribution on mirrors: $U_{ml}(x,y)=H_m H_l \exp\left[-\left(\frac{\pi}{L\lambda}\right)(x^2+y^2)\right]$

Normalized field distribution in the resonator anywhere:

$$U(x,y,z)=\frac{w_0}{w(z)}H_m\left[\frac{\sqrt{2}x}{w(z)}\right]H_l\left[\frac{\sqrt{2}y}{w(z)}\right]e^{-\frac{x^2+y^2}{w^2(z)}}\cdot e^{-i[kz-(1+m+l)\zeta(z)]}\cdot e^{-\left[ik\left(x^2+y^2\right)/2R(z)\right]}$$

complex radius of curvature: $\frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{\pi w^2(z)},$

$$H_0(x)=1,\quad H_1(x)=2x,$$
$$H_2(x)=2\left(2x^2-1\right),\cdots$$

$$\frac{1}{q(z)}=\frac{1}{z+i z_0} \longrightarrow$$

$$w(z)=w_0\left[1+(z/z_0)^2\right]^{1/2},\quad w_0=(\lambda z_0/\pi)^{1/2},$$
$$R(z)=z\left[1+(z_0/z)^2\right],\quad \zeta(z)=tg^{-1}(z/z_0).$$

z_0 is the Rayleigh distance. The confocal parameter: z_c = 2 z_0!

Properties of Gaussian beams

Properties of the TEM_{00} mode - intensity

Complex amplitude anywhere in the resonator:

$$U(\underline{r}) = U(\rho, z) = A_0 \frac{w_0}{w(z)} e^{-\frac{\rho^2}{w^2(z)}} \cdot e^{-i[kz-\zeta(z)]} \cdot e^{-[ik(\rho^2)/2R(z)]}, \quad \rho = (x^2 + y^2)^{1/2}$$

Intensity distribution: $I(\underline{r}) = |U(\underline{r})|^2$

$$I(\rho, z) = I_0 \left[\frac{w_0}{w(z)}\right]^2 exp\left(-\frac{2\rho^2}{w^2(z)}\right), \quad I_0 = |A_0|^2$$

Gaussian distribution at any z place, its maximum is on the beam axis (ρ = 0). The maximum value changes along the z axis:

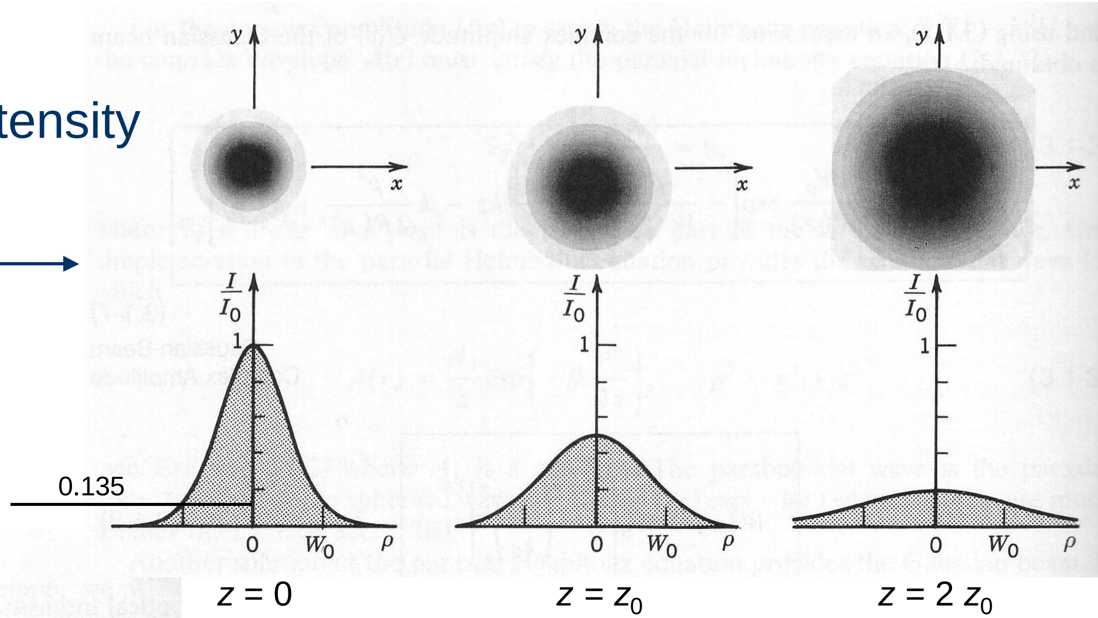
$$I(0, z) = I_0 \left[\frac{w_0}{w(z)}\right]^2 = \frac{I_0}{1 + (z/z_0)^2},$$

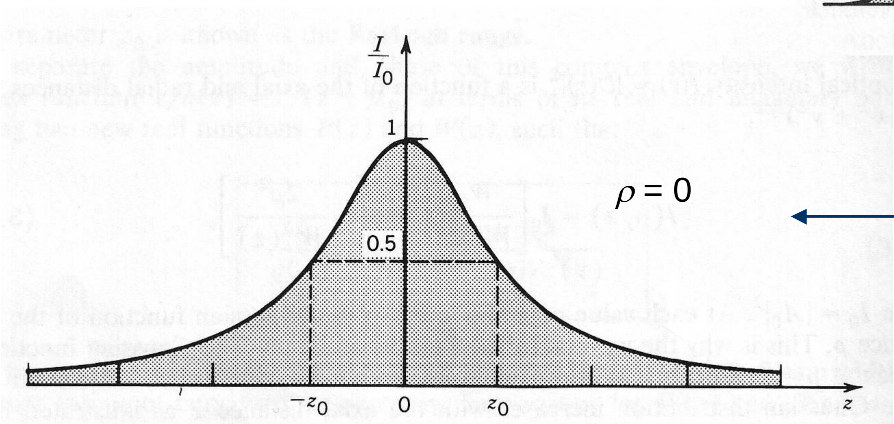
where I_0 is the maximum intensity at z = 0.

Properties of Gaussian beams

Properties of the TEM_{00} mode – intensity (cont.)

Normalized transversal beam intensity at different axial distances





Normalized beam intensity on the beam axis (maximum intensity). The maximum falls to the half at the Rayleigh distance (in confocal resonator on the mirrors).

Properties of Gaussian beams

Properties of the TEM_{00} mode – power

$$I(\rho,z)=I_0\left[\frac{w_0}{w(z)}\right]^2 exp\left(-\frac{2\rho^2}{w^2(z)}\right)$$

The total optical power carried by the beam:

$$P=\int_0^\infty I(\rho,z)2\pi\rho\,d\rho==2\pi I_0\left[\frac{w_0}{w(z)}\right]^2\int_0^\infty e^{-\frac{2\rho^2}{w^2(z)}}\rho\,d\rho$$

$$\int_0^\infty xe^{-cx^2}dx=\left|-\frac{1}{2c}e^{-cx^2}\right|_0^\infty=\frac{1}{2c},\quad c=\frac{2}{w^2(z)}$$

$$P=2\pi I_0\left[\frac{w_0}{w(z)}\right]^2\cdot\frac{w^2(z)}{4}$$

$$\boxed{P=\frac{1}{2}I_0\left(\pi w_0^2\right)}$$

Independent of z as it was expected (because of the factor $(w_0/w_z)^2$), the beam power is one-half the peak intensity times the beam area calculated with the radius of the beam waist.

Properties of Gaussian beams

Properties of the TEM_{00} mode – power (cont.)

The measurable quantity is P, it is useful to express I_0 in terms of P and rewrite expression of the intensity:

$$P = \frac{1}{2} I_0 \left(\pi w_0^2 \right) \quad \rightarrow \quad I_0 = \frac{2P}{\pi w_0^2}$$

$$I(\rho, z) = I_0 \left[\frac{w_0}{w(z)} \right]^2 exp\left(-\frac{2\rho^2}{w^2(z)} \right) = \frac{2P}{\pi w^2(z)} exp\left(-\frac{2\rho^2}{w^2(z)} \right),$$

$$I(0, z) = \frac{2P}{\pi w^2(z)}.$$

Anywhere the peak intensity can be calculated as the ratio of the beam power and the beam area. Measurement of the power with a finite size detector? How large is the error of the measurement?

Gauss-nyalábok tulajdonságai

Properties of the TEM_{00} mode – power measurement

ρ_0 is the radius of the detector, we calculate the error of the power measurement:

$$\frac{P(\rho_0)}{P} = \frac{1}{P}\int_0^{\rho_0} I(\rho,z)2\pi\rho\, d\rho = ?$$

$$\frac{P(\rho_0)}{P} = \frac{1}{P}\left[\int_0^{\infty} I(\rho,z)2\pi\,\rho\, d\rho - \int_{\rho_0}^{\infty} I(\rho,z)2\pi\,\rho\, d\rho\right] = 1 - e^{-\frac{2\rho_0^2}{w^2(z)}}.$$

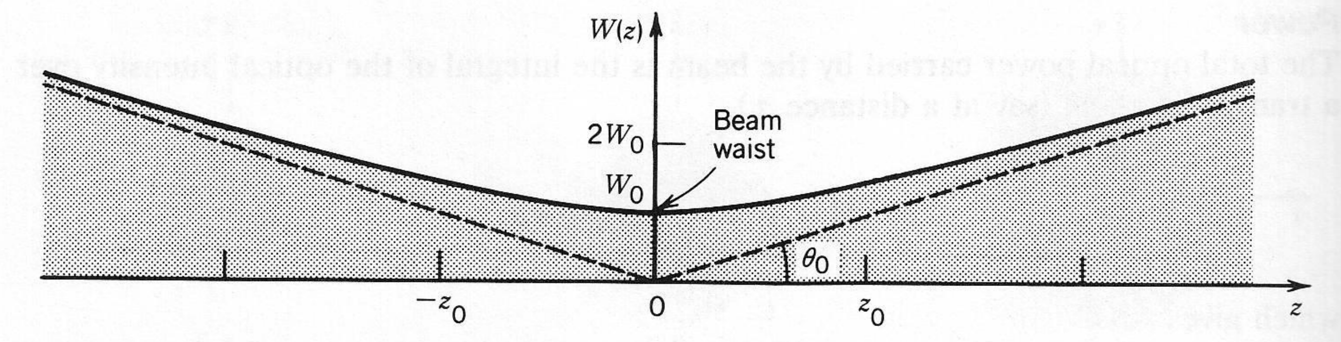
If $w(z) = \rho_0$, then $1-e^{-2}=0.86$ → we detect 86% of the total power.

If $1.5\,w(z)=\rho_0$, then $1-e^{-4.5}=0.99$ → we detect 99% of the total power. For an exact measurement $\rho_0 \geq 1.5\,w(z)$ is necessary (e.g. circular detector)!!

Properties of Gaussian beams

Properties of the *TEM*$_{00}$ mode – beam radius, beam divergence (rev.)

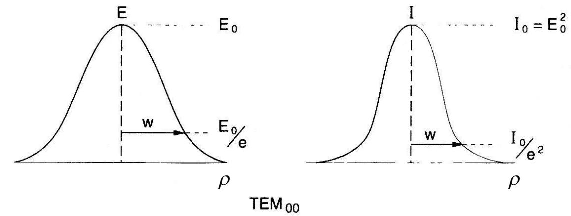
$$w(z) = w_0\left[1 + (z/z_0)^2\right]^{1/2}, \quad w(z=0) = w_0 = (\lambda z_0/\pi)^{1/2}, \quad z_0 = \frac{\pi w_0^2}{\lambda}.$$



At z_0 the beam radius is $\sqrt{2}\,w_0$ (in confocal resonator on the mirrors). w_0 is the radius of the beam waist. If $z >> z_0$

$$w(z) \approx \frac{w_0}{z_0} z = \theta z \quad \rightarrow \quad 2\theta = 2\frac{w_0}{z_0} = \frac{2}{\pi}\frac{\lambda}{w_0}.$$

divergence (full angle)



Properties of Gaussian beams

Divergence of the TEM_{00} beam ↔ divergence of an ideal plane wave

For a circular aperture of radius *a*: $1.22\dfrac{\lambda}{a}$

For a slit of width 2*a*: $1.0\dfrac{\lambda}{a}$

Gaussian-beam with waist w_0 : $0.64\dfrac{\lambda}{w_0}$ minimal!!!

Divergence of the multi-mode beam – M^2 factor

The radius of the beam is larger (Gaussian × Hermite or Laguerre polynomial)

$$W(z) = Mw(z) = M\frac{\lambda}{\pi w_0}z = M\frac{\lambda}{\pi\frac{W_0}{M}} = M^2\frac{\lambda}{\pi W_0}z$$

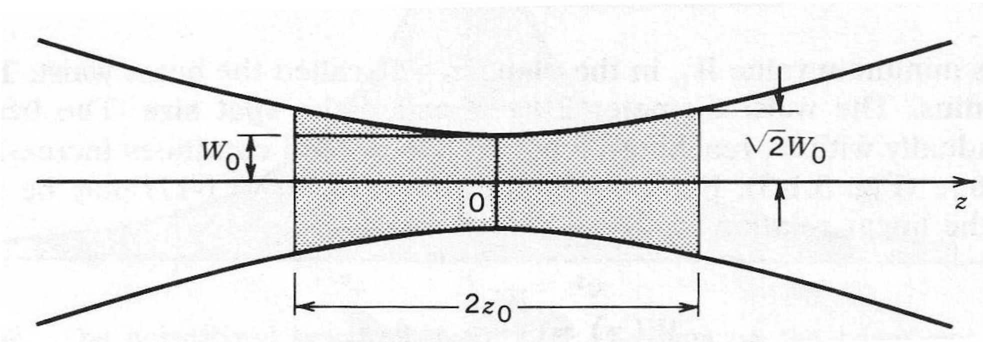
$$2\theta = M^2\frac{2}{\pi}\frac{\lambda}{W_0}$$

M^2 characterizes also the deviation from the ideal Gaussian beam with waist W_0!

Properties of Gaussian beams

Properties of the TEM_{00} mode – depth of focus

Since the beam has its minimum width at z=0, its best focus is at the plane z=0. In Rayleigh distance from z=0 the beam radius grows by a factor of $\sqrt{2}$. The depth of focus is two times the Rayleigh distance and is equal with the confocal parameter z_c.



$$2z_0 = \frac{2\pi w_0^2}{\lambda}.$$

The depth of focus is directly proportional to the area of the beam at its waist, and inversely proportional to the wavelength.

When a beam is focused to a small spot size, the depth of focus is short and the plane of focus must be located with greater accuracy. A small spot size and a long depth of focus cannot be obtained simultaneously.

Properties of Gaussian beams

Properties of the TEM_{00} mode – depth of focus (cont.)

E.g. He-Ne laser λ = 633 nm

$2\,w_0 = 2\ cm$ $$2z_0 = \frac{2\pi w_0^2}{\lambda} = \frac{2\pi \cdot 10^{-4}}{6.33 \cdot 10^{-7}} m \approx 1 km!$$

$2\,w_0 = 2\ \mu m$ $$2z_0 = \frac{2\pi w_0^2}{\lambda} = \frac{2\pi \cdot 10^{-12}}{6.33 \cdot 10^{-7}} m \approx 10\ \mu m!$$

Properties of the TEM_{00} mode – phase

$$U(\underline{r}) = U(\rho, z) = A_0 \frac{w_0}{w(z)} e^{-\frac{\rho^2}{w^2(z)}} \cdot e^{-i[kz - \zeta(z)]} \cdot e^{-[ik(\rho^2)/2R(z)]}, \quad \rho = \left(x^2 + y^2\right)^{1/2}$$

$$\varphi(\rho, z) = kz - \zeta(z) + \frac{k\rho^2}{2R(z)},$$

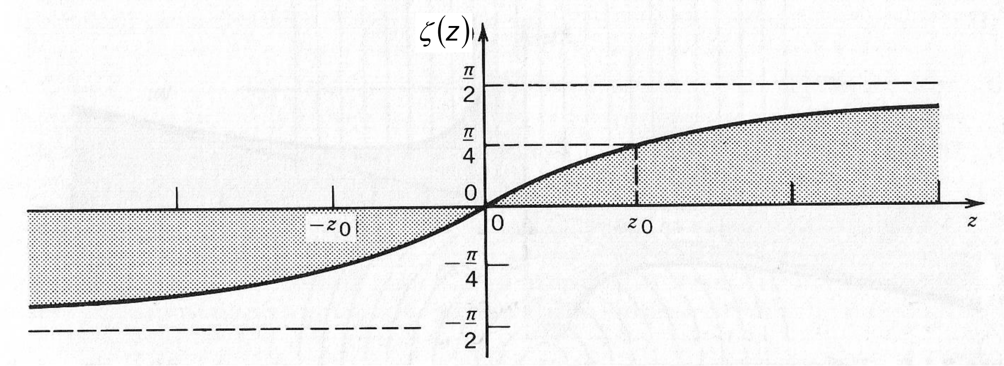
on the optical axis $$\varphi(0, z) = kz - \zeta(z).$$

Properties of Gaussian beams

Properties of the TEM_{00} mode – phase (cont.)

on the optical axis $\varphi(0,z) = kz - \zeta(z), \quad \zeta(z) = tg^{-1}(z/z_0).$

phase of a plane wave



$\zeta(z)$ is the excess delay of the wavefront compared to a plane or a spherical wave.

The total accumulated excess retardation from $z = -\infty$ to $z = \infty$ is π !
(Guoy effect)

Properties of Gaussian beams

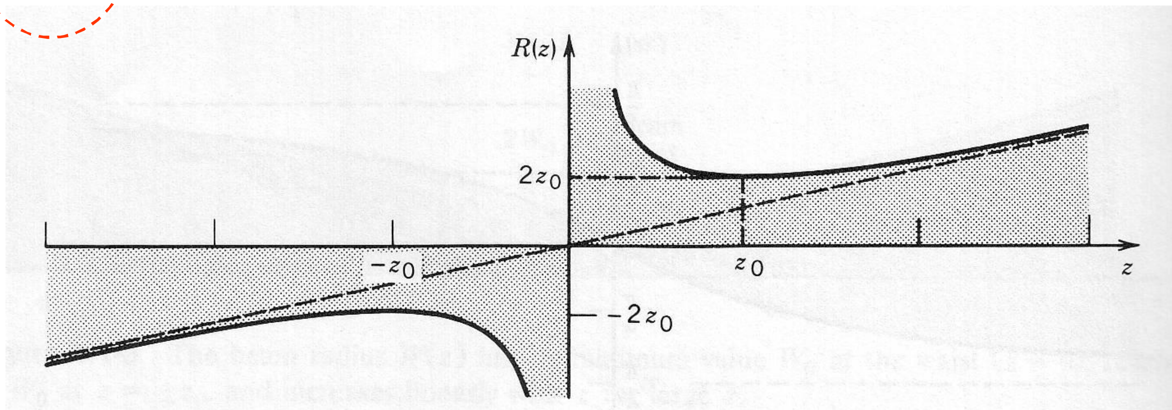
Properties of the TEM_{00} mode – wavefronts

$$\varphi(\rho,z)=kz-\zeta(z)+\frac{k\rho^2}{2R(z)}$$

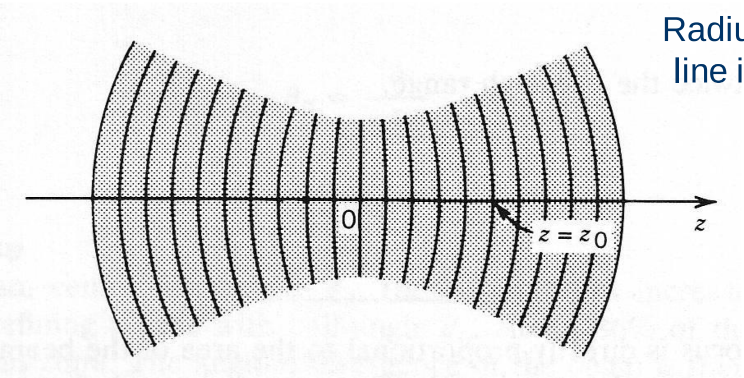
responsible for the wavefront bending at off-axis points

$$R(z)=z\left[1+(z_0/z)^2\right]$$

Minimum value of $R(z)$ is $2z_0$ at z_0.



Radius of curvature of the wavefronts of a Gaussian beam. The dashed line is the radius of curvature of a spherical wave. For $z >> z_0$ they are equal.



Wavefronts of a Gaussian beam, the wavefronts are plane in the middle (z=0) and in the infinity.

Properties of Gaussian beams

Exercises

1. Stability of resonators
   A resonator has two mirrors with radius of curvature 1 m and 1.5 m, resp. Determine the length *L* of the stable operation?

$$0 \le g_1 g_2 \le 1, \quad g_{1,2} = 1 - \frac{L}{R_{1,2}}, \quad 0 \le (1-L)\left(1 - \frac{2}{3}L\right) \le 1$$

$$0 \le (1-L)\left(1 - \frac{2}{3}L\right) \quad \rightarrow \quad L \le 1, \quad L \ge \frac{3}{2}$$

$$1 - \frac{2}{3}L - L + \frac{2}{3}L^2 \le 1$$

$$L\left(\frac{2}{3}L - \frac{5}{3}\right) \le 0 \quad \rightarrow \quad L \ge 0, \quad L \le \frac{5}{2}$$

$$\boxed{0 \le L \le 1, \quad \frac{3}{2} \le L \le \frac{5}{2}}$$

Properties of Gaussian beams

Exercises

2. Parameters of Gaussian beam

The TEM_{00} mode beam of a 1 mW He-Ne laser (wavelength of 633 nm) has a spot diameter $2\ w_0 = 0.1$ mm.

a) Determine the angular divergence of the beam, its depth of focus, its diameter at $z=3.5\cdot10^5$ km (~ the distance to the moon).

b) What is the radius of curvature of the wavefront at $z = 0$, $z = z_0$, $z = 2\ z_0$?

c) What is the maximum optical intensity (in W/cm^2) at the beam center ($z = 0$, $\rho = 0$) and at the axial point $z = z_0$? Compare this with the intensity at $z = z_0$ of a 100 W spherical wave produced by a small point source at $z = 0$!

$$z_0 = \frac{\pi w_0^2}{\lambda}, \quad 2\theta_0 = \frac{2}{\pi}\frac{\lambda}{w_0}, \quad 2z_0, \quad w(z) \approx \frac{w_0}{z_0} z = \theta_0 z, \quad R(z) = z\left[1 + (z_0/z)^2\right],$$

$$I(0,z) = \frac{2P}{\pi w^2(z)}, \quad I_{\text{point source}} = \frac{P}{4\pi z^2}$$

Properties of Gaussian beams

Exercises

3. Power of the Gaussian beam

   The beam waist of an Ar-ion laser ($\lambda = 488$ nm) is 1 mm (outside the laser, $2w_0$). You measure a power of 2 W with a circular detector (diameter 3 mm) at 10 m from the beam waist. Determine the power of the beam!

4. Equivalent resonator

   The resonator with a plane and a concave mirror (radius of curvature 1 m) has a length of 0.3 m. Determine the divergence of the beam when the output mirror is the plane mirror and the wavelength is 1 µm?

   Steps: half symmetric resonator $\rightarrow$ symmetric $\rightarrow z_0 \rightarrow w_0$ on the plane mirror $\rightarrow 2\theta$

$$z_c^2 = (2R - L)L, \quad z_c = 2z_0$$