Topological Insulators 2 (Topological superconductors)
2020 Spring, Lecture 5
The Pfaffian as the topological invariant of 0D and 1D superconductors (v1)

(Dated: April 1, 2020)

Log v1→v2: Hint added to exercise 1. Comment added to exercise 3/4.

I. CONTROL QUESTIONS FOR "HAMILTONIANS, TOPOLOGY AND SYMMETRY"

1. Consider a spin-z-conserving Hamiltonian for spin-1/2 particles on two sites:

$$\hat{H} = E_L(|L\uparrow\rangle\langle L\uparrow| - |L\downarrow\rangle\langle L\downarrow|) + E_R(|R\uparrow\rangle\langle R\uparrow| - |R\downarrow\rangle\langle R\downarrow|) + v(|L\uparrow\rangle\langle R\uparrow| + |L\downarrow\rangle\langle R\downarrow| + h.c.), \quad (1)$$

with real parameters $0 < v < E_L < E_R$. Now change E_R to $-E_R$. Is this change a topological phase transition or not? (Hint: is there a level that crosses zero energy along the way?)
A) Not topological.
B) Topological.
C) Depends on the value of v.
D) This type of change is forbidden by spin-z conservation law.

Answer: B) The z component of the spin is conserved, so first we should consider the $\uparrow$ block and the $\downarrow$ block separately. For both blocks, there will be an energy eigenvalue crossing zero energy, and this satisfies the definition of topological phase transition for a zero-dimensional system.

2. List all symmetries mentioned in the chapter, and categorize them! Note that there are multiple ways to categorize.

Answer: A few ways to categorize them:

1) unitary/antiunitary,

2) maps the Hamiltonian H to itself / to $-H$

3) generically, are there "gap closings" at zero energy (e.g., no symmetry), or there's level repulsion instead (e.g., sublattice symmetry)

3. Which statements are true?
1) The Pfaffian is a function that maps any square matrix to a complex number. FALSE, since it's defined only for anti-symmetric matrices.
2) If you square the Pfaffian of a matrix, then you get the determinant of that matrix. TRUE.
3) The BdG matrix is always antisymmetric. FALSE, but it's true that any BdG matrix can be transformed to an antisymmetric matrix with a specific unitary transformation, the so-called 'Majorana transformation'.
4) The Pfaffian is defined as the square root of the determinant. FALSE. The Pfaffian is defined as a polynomial of the matrix elements, see wikipedia. The definition 'square root of the determinant' cannot be a equivalent definition with the original one, since it is insufficient to serve as a definition: e.g., if the determinant is -1, then it's square root can be i or $-i$.

4. Find the typo in the last display equation of "The Pfaffian invariant". (Find the related typos in the display equations of the chapter "Bulk-edge correspondence in the Kitaev chain" as well.)

Answer: In the formulas containing the Pfaffian, often it's $H_{\rm BdG}$ in the argument of Pf(). But $H_{\rm BdG}$ is not antisymmetric, so it does not have a Pfaffian. So it should be replaced by $\tilde{H}_{\rm BdG}$, which is antisymmetric so it has a Pfaffian.

5. An extremely simple zero-dimensional superconducting system is described by the BdG Hamiltonian $H_{\rm BdG} = \sigma_z$. Another one is described by $H_{\rm BdG} = -\sigma_z$. Can we deform the two cases into each other such that the gap remains open and we have particle-hole symmetry all along the way?

Answer: No. One way to argue is that it's easy to reconstruct the Fock-space Hamiltonian from this BdG Hamiltonian, and follow the true ground state.

6. Consider a quantum dot with a single energy level, with spin restricted to "up", proximitized by a superconductor, with magnetic field present. The Hamiltonian for this can be constructed using only one fermionic operator $\hat{c}$, using complex numbers, taking products, adjoints, sum. How many topologically different Hamiltonians can be constructed? If your answer is 2 or more than 2, give examples.
   A) 0: no Hamiltonian that makes sense as a superconductor.
   B) 1: many different Hamiltonians, same topological invariant.
   C) 2.
   D) more than 2.

7. What is the Pfaffian associated to the BdG Hamiltonian $H_{\mathrm{BdG}} = \sigma_z$? What is the Pfaffian associated to $H_{\mathrm{BdG}} = -\sigma_z$? Wikipedia's Pfaffian article might be helpful.

II. CONTROL QUESTIONS FOR "BULK-EDGE CORRESPONDENCE IN THE KITAEV CHAIN"

1. Assume that a BdG quasiparticle is its own particle-hole partner. Does that imply that it is a zero mode?

2. Using the first formula of the chapter, express γ_1 and γ_2 as functions of c and $c^\dagger$.

3. Check that the expression of the single-site Hamiltonian with the Majorana operators in the text is correct.

4. In the last display equation of the section 'Majorana modes appearing at a domain wall between different phases', two wave functions are given, one with a '+' sign and one with a '-'. Which one is the physical one?

5. What is the set of "allowed" momenta for a 3-site chain with periodic boundary conditions? What is this set in the case of antiperiodic boundary conditions?

6. What differences do you observe between the notation of the first 4 lectures and this one?

   Answer:

   1) Momentum basis uses a different sign.
   2) We defined Majorana (zero) modes as their on particle-hole partner; here, they "define" it as equal-weight superposition of particle and hole.