Experimental realization of Majorana zero modes

April 14, 2020

Proximity effect

If a material in the superconducting phase is in close contact to a normal material then superconductivity can "spill over" to the normal material, this is called the superconducting proximity effect. Consider the Hamiltonian

$$H = \sum_{ij} h^N_{ij} c^\dagger_i c_j + \sum_{ij} h^S_{ij} b^\dagger_i b_j + \sum_{ij} \left(\Delta_{ij} b^\dagger_i b^\dagger_j + h.c. \right) + \sum_{ij} \left(\Gamma_{ij} b^\dagger_i c_j + h.c. \right), \tag{1}$$

here the c_i operators act in the normal system, b_i operators act in the superconductor, the term involving Γ_{ij}-s corresponds to the coupling between the two. The BdG matrix is then:

$$\mathcal{H} = \begin{pmatrix} h_N & 0 & \Gamma & 0 \\ 0 & -h^*_N & 0 & -\Gamma^* \\ \Gamma^\dagger & 0 & h_S & \Delta \\ 0 & -\Gamma^{\dagger *} & -\Delta^* & -h^*_S \end{pmatrix} \tag{2}$$

where we organize the Nambu spinor according to $\begin{pmatrix} c & c^\dagger & b & b^\dagger \end{pmatrix}^T$. A schematic representation of this system and the spectrum is depicted in Fig. 1. As the schematic spectrum shows one can expect that due to the presence of the superconductor the normal region will also develop a superconducting gap. We can, using quasi-degenerate perturbation theory, derive an effective BdG matrix

$$\mathcal{H}^{Eff} = \begin{pmatrix} h'_N & \Delta' \\ -\Delta'^* & -h'^*_N \end{pmatrix} \tag{3}$$

where we only act on the Nambu spinor $\begin{pmatrix} c & c^\dagger \end{pmatrix}^T$.

In the appendix we show, in a simple model, that if the pair potential was $\propto b^\dagger_{i\uparrow} b^\dagger_{i\downarrow}$ in the bulk then the effective pair potential Δ' will inherit this structure after this step, also that the magnitude of the induced gap is inversely proportional to the bulk gap and is proportional to the square of the coupling strength. In the next chapter we investigate how topological superconductivity can be engineered with the help of proximity induced superconductivity.

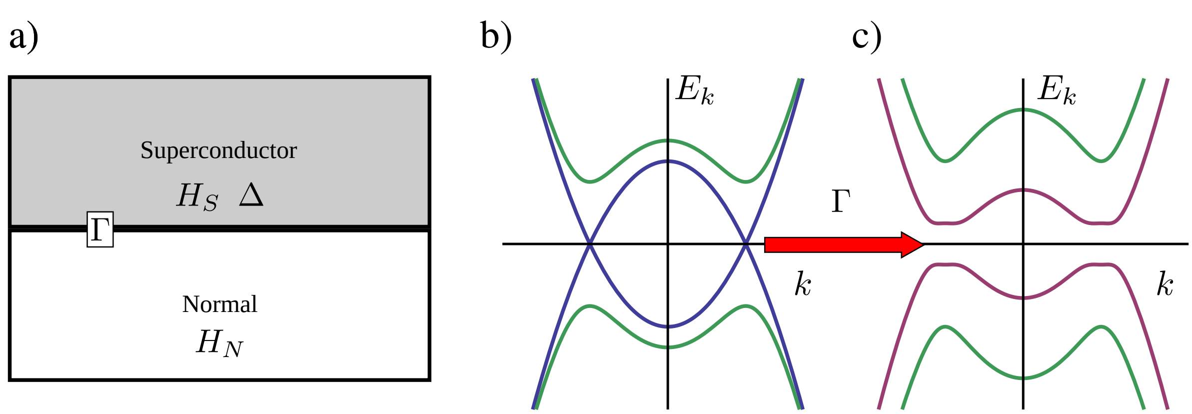


Figure 1: Proximity effect. a) Geometric representation of a superconductor in close contact with a normal metal. b) A schematic representation of the BdG spectrum of the whole system without taking the Γ hopping terms in to account. Green solid lines are the states of the superconductor blue lines represent the metal. c) the spectrum of the joined system after taking Γ in to account.

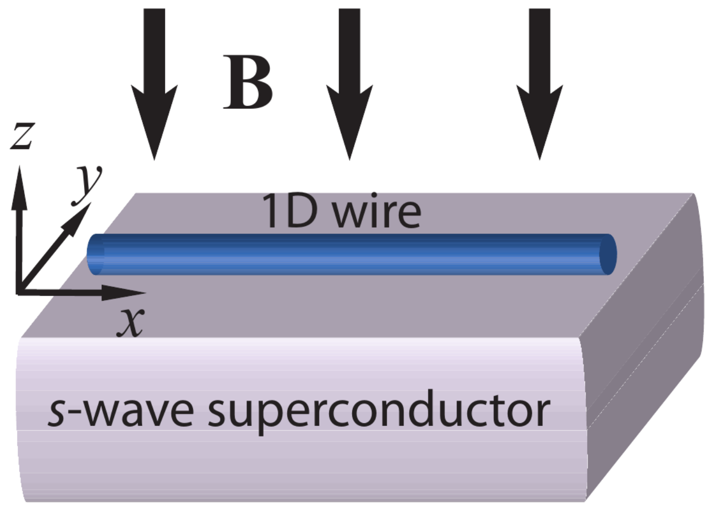


Figure 2: The geometrical setup realizing the Lutchyn wire model. A nanowire with spin-orbit coupling and proximitized by a bulk s-wave superconductor subject to an external magnetic filed.

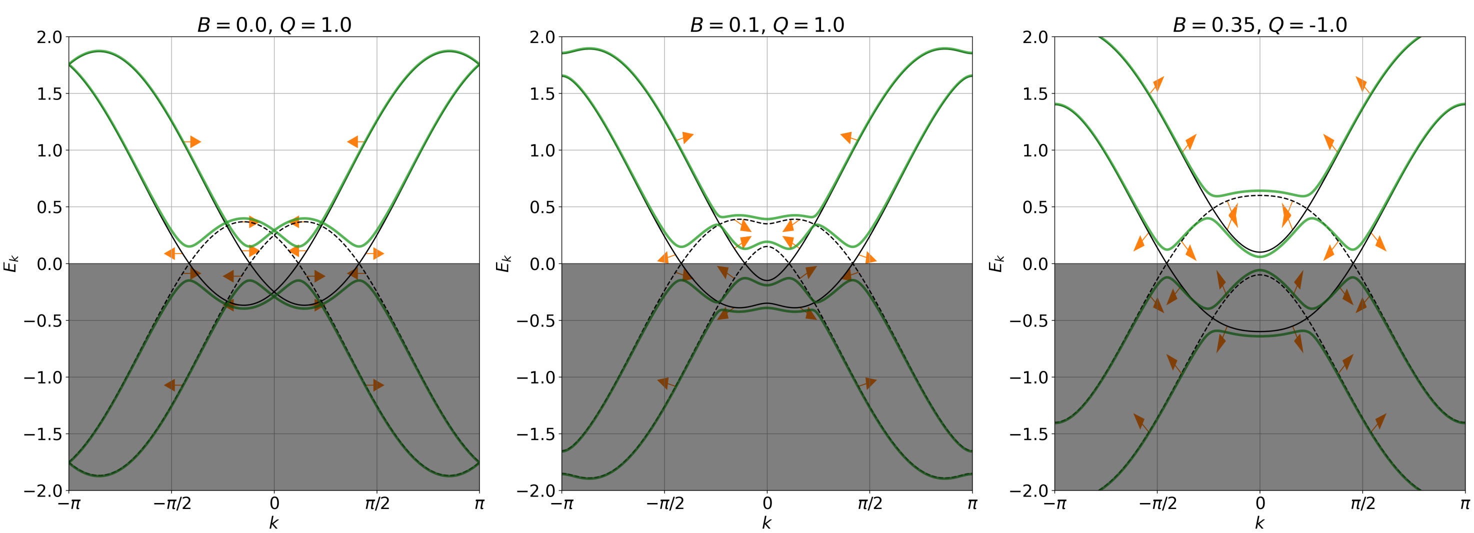


Figure 3: Spectrum and spin pattern of the Lutchyn wire model. Black solid and dashed curves correspond to $\Delta = 0$ with $\mu = -0.75$ and $\alpha = 0.5$. Yellow arrows depict the expectation value of the y(horizontal) and z(vertical) spin components for a given state. The green solid line depicts the spectrum for $\Delta = 0.15$. The topological invariant Q is calculated by evaluating the product of the Pfaffians of the BdG matrix at $k = 0$ and at $k = \pi$.

Lutchyn wire on a lattice - calculating the topological invariant

A lattice realization of the Lutchyn wire depicted in Fig. 2 can be written as

$$\begin{aligned} H_{\text{Lutchyn}} &= \sum_m \begin{pmatrix} c^\dagger_{m\uparrow} & c^\dagger_{m\downarrow} \end{pmatrix} \begin{pmatrix} B_z - \mu & 0 \\ 0 & -B_z - \mu \end{pmatrix} \begin{pmatrix} c_{m\uparrow} \\ c_{m\downarrow} \end{pmatrix} \\ &- \frac{1}{2} \sum_m \left[\begin{pmatrix} c^\dagger_{m+1\uparrow} & c^\dagger_{m+1\downarrow} \end{pmatrix} \begin{pmatrix} t & \alpha \\ -\alpha & t \end{pmatrix} \begin{pmatrix} c_{m\uparrow} \\ c_{m\downarrow} \end{pmatrix} + \text{h.c.} \right] \\ &- \Delta \sum_m \left[c^\dagger_{m\uparrow} c^\dagger_{m\downarrow} + \text{h.c.} \right]. \end{aligned} \tag{4}$$

That is we have a Zeeman term in parallel to the z direction, we have normal hopping t along the wire, a spin orbit coupling term proportional to $\mathrm{i}\alpha\sigma_y$ and s-wave type, onsite pairing term with magnitude Δ. The corresponding BdG matrix in momentum space is

$$\mathcal{H}_{\text{Lutchyn}} = \begin{pmatrix} -t\cos(k) + B_z - \mu & -\mathrm{i}\alpha\sin(k) & 0 & -\Delta \\ \mathrm{i}\alpha\sin(k) & -t\cos(k) - B_z - \mu & \Delta & 0 \\ 0 & \Delta & t\cos(k) - B_z + \mu & \mathrm{i}\alpha\sin(k) \\ -\Delta & 0 & -\mathrm{i}\alpha\sin(k) & t\cos(k) + B_z + \mu \end{pmatrix}. \tag{5}$$

This is now an object defined in the whole BZ, thus we can calculate the topological invariant of the system by calculating the Pfaffian at $k = 0$ and at $k = \pi$ as was done by Jay Sau on topocondmat (chapter "Bulk-edge correspondence in the Kitaev chain"). First we need to transform $\mathcal{H}_{\text{Lutchyn}}$ to the Majorana basis as $\mathcal{H}'_{\text{Lutchyn}} = U\mathcal{H}_{\text{Lutchyn}}U^\dagger$, where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & -\mathrm{i} \end{pmatrix}. \tag{6}$$

The transformed BdG matrix is

$$\mathrm{i}\mathcal{H}'_{\text{Lutchyn}} = \begin{pmatrix} 0 & 0 & B_z - t\cos(k) - \mu & \Delta - \mathrm{i}\alpha\sin(k) \\ 0 & 0 & -\Delta + \mathrm{i}\alpha\sin(k) & -B_z - t\cos(k) - \mu \\ -B_z + t\cos(k) + \mu & \Delta + \mathrm{i}\alpha\sin(k) & 0 & 0 \\ -\Delta - \mathrm{i}\alpha\sin(k) & B_z + t\cos(k) + \mu & 0 & 0 \end{pmatrix} \tag{7}$$

Thus the topological invariant for this system reads

$$\begin{aligned} Q &= \text{sign}\left(\text{Pf}\left[\mathrm{i}\mathcal{H}'_{\text{Lutchyn}}(0)\right] \text{Pf}\left[\mathrm{i}\mathcal{H}'_{\text{Lutchyn}}(\pi)\right]\right) \\ &= \text{sign}\left[\left(B_z^2 - \Delta^2 - (t+\mu)^2\right)\left(B_z^2 - \Delta^2 - (t-\mu)^2\right)\right]. \end{aligned} \tag{8}$$

The spectrum, spin structure and topological invariant of the model is depicted in Fig. 3 for various system parameters.
<u>Task to tackle at home:</u>
Taking the EFA for $k = 0$ we have

$$\begin{aligned} \mathcal{H}^{\text{EFA}}_{\text{Lutchyn}} &= \begin{pmatrix} \frac{\hat{p}^2}{2m} + B_z - \mu' & -\mathrm{i}\alpha\hat{p} & 0 & -\Delta \\ \mathrm{i}\alpha\hat{p} & \frac{\hat{p}^2}{2m} - B_z - \mu' & \Delta & 0 \\ 0 & \Delta & -\frac{\hat{p}^2}{2m} - B_z + \mu' & \mathrm{i}\alpha\hat{p} \\ -\Delta & 0 & -\mathrm{i}\alpha\hat{p} & -\frac{\hat{p}^2}{2m} + B_z + \mu \end{pmatrix} \end{aligned} \tag{9}$$

$$= \left[\left(\frac{\hat{p}^2}{2m} - \mu\right)\sigma_0 + B_z\sigma_z + \alpha\hat{p}\sigma_y\right] \otimes \tau_z + \Delta\sigma_y \otimes \tau_y \tag{10}$$

where we introduced $m = 1/t$ and redefined the chemical potential as $\mu' = t + \mu$.

1. Calculate coefficients for Andeev and inverse Andreev reflection from the EFA matrix. Show that the relative sign of the reflection coefficients is the topological invariant calculated above. And show that under the right circumstances the EFA can be mapped to the Kitaev wire!

2. Calculate the phase diagram in terms of B_z, Δ and μ.

3. Find Majoranas from the continuum or from the TB model of fixed system size at the interface of a trivial and topological superconductor phase.

Delft experiment

What could go wrong:

- Higher magnetic field allows for a wider range in μ for the topological phase. However higher magnetic filed tends to align particles and holes in the same direction, thus it inhibits pairing.
- High magnetic field also has detrimental effects on the gap itself.
- Orientation of the magnetic field is important! If we align the magnetic field with the effective field of the spin-orbit interaction, then it will not split the spectrum at $k = 0$, thus the Fermi surface will have two fermionic species.
  Task to tackle at home: generalize the models of the previous for any magnetic field orientation and check if statement is true!
- We only focused on a single modded wire! In actual experiments the nanowires have a finite width and thus potentially more than one mode, thus making the system more complicated. The signatures of Majorana bound states *e.g.* ZBCP could be influenced by the presence of additional modes.

Take home messages

- We have seen that a topological superconducting phase can be realized even if the pair potential is not like it was in the Kitaev model *i.e.* $\propto c^\dagger_{m+1}c^\dagger_m$ but as it was in a conventional superconductor *i.e.* $\propto c^\dagger_{m\uparrow}c^\dagger_{m\downarrow}$.
- The key ingredients we needed were spin-orbit coupling and an external magnetic field.
- In the emerging topological phase the Fermi surface will have only a single fermion species. That is the spectrum of particle like excitations will have a single left moving and a single right moving state. This is the key similarity between the investigated model and Kitaev's idealized model.
- There is a growing consensus that this phase was observed in Delft as reported in 2012 and in 2018.

Appendix

We give a short derivation of a proximity induced superconductivity in a simple model system.

Consider the BdG matrix

$$\mathcal{H}_{BdG} = \begin{pmatrix} \overbrace{\begin{bmatrix} \varepsilon_1 & 0 & & \\ 0 & \varepsilon_1 & & \\ & & -\varepsilon_1 & 0 \\ & & 0 & -\varepsilon_1 \end{bmatrix}}^{N} & \overbrace{\begin{bmatrix} t & & & \\ & t & & \\ & & -t & \\ & & & -t \end{bmatrix}}^{\tilde{\Gamma}} \\ \begin{bmatrix} t & & & \\ & t & & \\ & & -t & \\ & & & -t \end{bmatrix} & \underbrace{\begin{bmatrix} \varepsilon_2 & 0 & 0 & \Delta \\ 0 & \varepsilon_2 & -\Delta & 0 \\ 0 & -\Delta & -\varepsilon_2 & 0 \\ \Delta & 0 & 0 & -\varepsilon_2 \end{bmatrix}}_{S} \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} c_\uparrow \\ c_\downarrow \\ c^\dagger_\uparrow \\ c^\dagger_\downarrow \\ b_\uparrow \\ b_\downarrow \\ b^\dagger_\uparrow \\ b^\dagger_\downarrow \end{pmatrix}. \tag{11}$$

This model might represent a spinfull qunatum dot next to a single site superconducting island, or making the ε_i-s momentum dependent it could also represent an extended system.

In the first step we diagonalize the superconductor $S \to S' = W^\dagger S W$ this also entails the transformation of the coupling $\tilde{\Gamma} \to \tilde{\Gamma}' = \tilde{\Gamma} W$. The matrix of the transformation is given by

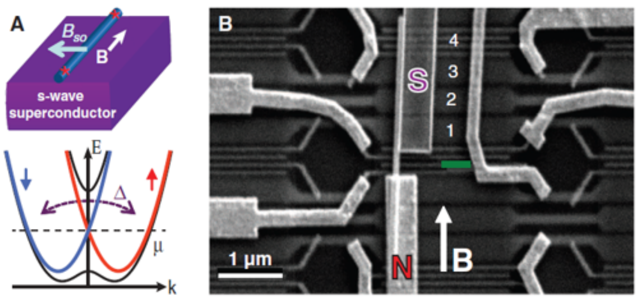
$$W = \begin{pmatrix} \cos(\varphi/2) & 0 & 0 & -\sin(\varphi/2) \\ 0 & \cos(\varphi/2) & \sin(\varphi/2) & 0 \\ 0 & -\sin(\varphi/2) & \cos(\varphi/2) & 0 \\ \sin(\varphi/2) & 0 & 0 & \cos(\varphi/2) \end{pmatrix} \tag{12}$$

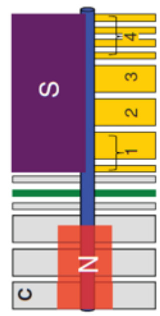
with $\cos(\varphi) = \frac{\varepsilon_2}{\sqrt{\varepsilon_2^2+\Delta^2}}$. Next we apply second order quasi-degenerate perturbation theory, assuming $\frac{\varepsilon_1}{\sqrt{\varepsilon_2^2+\Delta^2}}$ is the small parameter in the model. We obtain the effective BdG describing the N region as

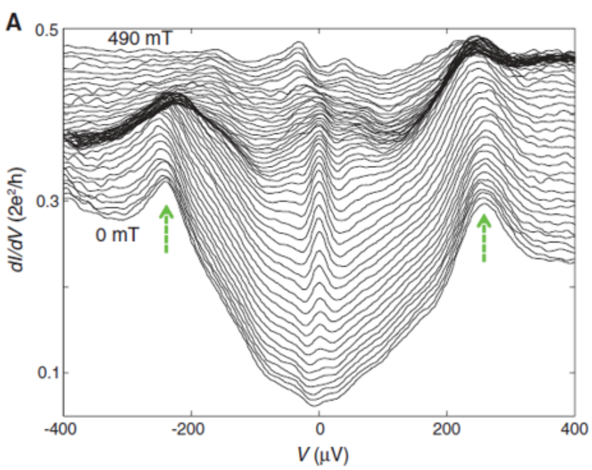
$$N' \approx \begin{pmatrix} \varepsilon_1 & 0 & & \\ 0 & \varepsilon_1 & & \\ & & -\varepsilon_1 & 0 \\ & & 0 & -\varepsilon_1 \end{pmatrix} + \frac{1}{\varepsilon_2^2+\Delta^2-\varepsilon_1^2} \begin{pmatrix} -(\varepsilon_1+\varepsilon_2)t^2 & & & t^2\Delta \\ & -(\varepsilon_1+\varepsilon_2)t^2 & -t^2\Delta & \\ & -t^2\Delta & (\varepsilon_1+\varepsilon_2)t^2 & \\ t^2\Delta & & & (\varepsilon_1+\varepsilon_2)t^2 \end{pmatrix} \tag{13}$$

that is the induced gap is $\propto \frac{t^2}{\Delta}$, and it's structure is inherited from the bulk superconductor.

Delft experiments

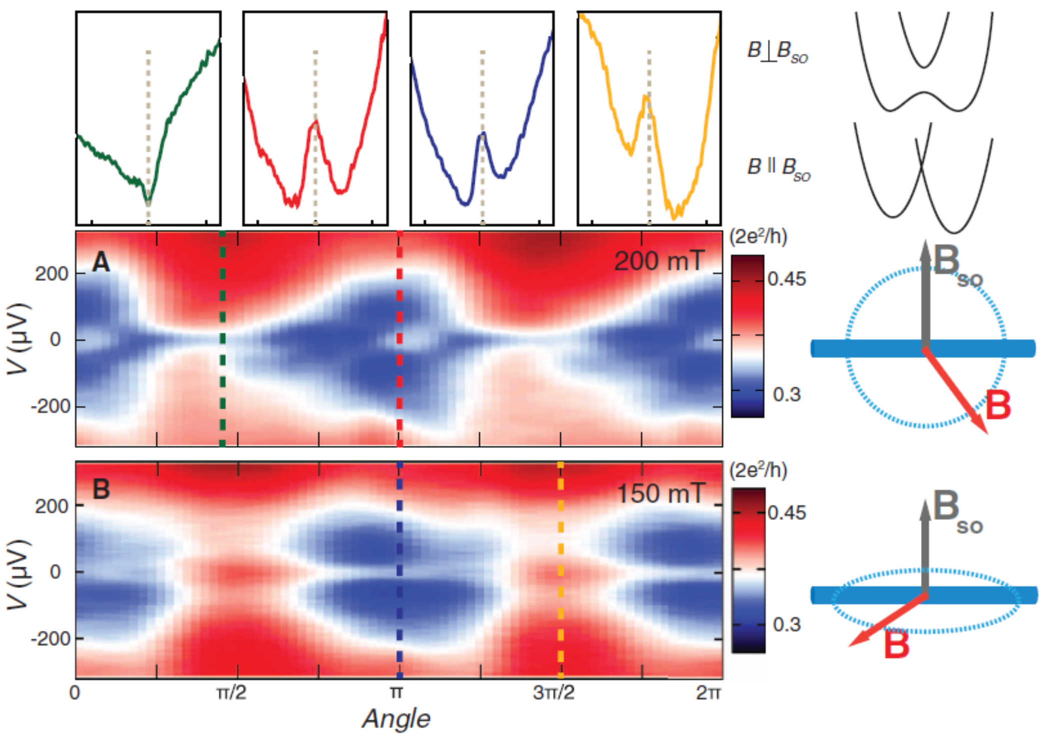




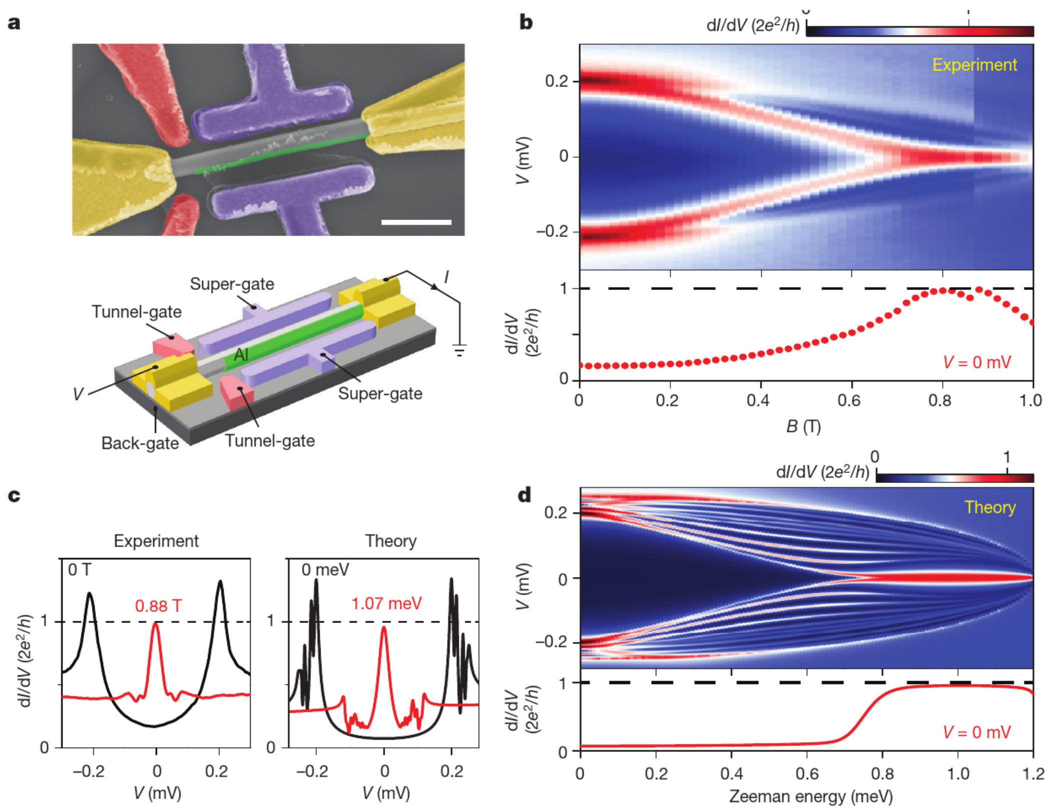


V. Mourik *et al.* Science, 336, 1003 (2012)

Aligning the magnetic field with the spin-orbit coupling field kills ZBCP



Quantized ZBCP



Zhang et al. Nature **556**, 74 (2018)

ZBCP due to Andreev bound states