Experimental signatures of Majorana fermions: Zero-bias conductance peak

April 8, 2020

Quasiparticles at the NS junction

Recap of BdG

A generic Fock space Hamiltonian of a superconductor is recast with the help of the BdG trick as

$$\begin{aligned} H &= \sum_{ij} h_{ij} c_i^\dagger c_j + \Delta_{ij} c_i^\dagger c_j^\dagger + \text{h.c.} \qquad (1) \\ &= \frac{1}{2} \begin{pmatrix} \boldsymbol{c}^\dagger & \boldsymbol{c} \end{pmatrix} \underbrace{\begin{pmatrix} \boldsymbol{h} & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^* & -\boldsymbol{h}^* \end{pmatrix}}_{\mathcal{H}_{\text{BdG}}} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix} + \frac{1}{2} \text{Tr}[\boldsymbol{h}] \mathbb{I}, \end{aligned}$$

where we introduced the Nambu spinor $\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}^\dagger \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_1^\dagger \\ \vdots \end{pmatrix}$ built from creation and annihilation operators. Remember that hermiticity of H requires $\boldsymbol{h} = \boldsymbol{h}^\dagger$ and $\boldsymbol{\Delta} = -\boldsymbol{\Delta}^\top$. The positive eigenvalues of the BdG matrix give the excitation spectrum.

$$\mathcal{H}_{\text{BdG}} \psi_n = E_n \psi_n \qquad (2)$$

The BdG trick forces PHS on $\mathcal{H}_{BdG}$ this is not physical it is built in the formalism. PHS is represented

$$\mathcal{P} = \sigma_x \mathcal{K}_R \qquad (3)$$

where σ_x is the appropriate Pauli matrix in Nambu space and the operator $\mathcal{K}_R$ is complex conjugation in real space. The effect of PHS is

$$\mathcal{P} \mathcal{H}_{\text{BdG}} \mathcal{P}^{-1} = -\mathcal{H}_{\text{BdG}} \qquad (4)$$

p-wave and s-wave models in k-space

We have so far investigated the Kitaev wire extensively, this shell be our p-wave model:

$$H_{\text{Kitaev}} = -\sum_m \mu c_m^\dagger c_m - \sum_m \left(v c_m^\dagger c_{m+1} + \text{h.c.} \right) - \sum_m \left(\Delta c_{m+1}^\dagger c_m^\dagger + \text{h.c.} \right) = \frac{1}{2} \sum_k f_k^\dagger \mathcal{H}_{\text{Kitaev}}(k) f_k + \text{const} \qquad (5)$$

Where in the last step we performed first the BdG trick and than a Fourier transform in the Nambu basis

$$f_m = \begin{pmatrix} c_m \\ c_m^\dagger \end{pmatrix} \to f_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = \sum_m f_m e^{\mathrm{i}km}. \qquad (6)$$

Thus the BdG matrix for the Kitaev model reads

$$\mathcal{H}_{\text{Kitaev}}(k) = \begin{pmatrix} -\mu - 2v\cos(k) & -2\mathrm{i}\sin(k)\,\Delta \\ 2\mathrm{i}\sin(k)\,\Delta^* & \mu + 2v\cos(k) \end{pmatrix}. \qquad (7)$$

For the s-wave case we

$$H_{\text{s-wave}} = -\sum_{m\sigma} \mu c^\dagger_{m\sigma} c_{m\sigma} - \sum_{m\sigma} \left(v c^\dagger_{m\sigma} c_{m+1\sigma} + \text{h.c.} \right) - \sum_{m} \left(\Delta c^\dagger_{m\uparrow} c^\dagger_{m\downarrow} + \text{h.c.} \right) = \frac{1}{2} \sum_k f^\dagger_k \mathcal{H}_{\text{Full s-wave}}(k) f_k + \text{const}, \tag{8}$$

here the Nambu spinor has four component, and thus the BdG matrix will also be 4×4

$$f_m = \begin{pmatrix} c_{m\uparrow} \\ c_{m\downarrow} \\ c^\dagger_{m\uparrow} \\ c^\dagger_{m\downarrow} \end{pmatrix} \to f_k = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \\ c^\dagger_{-k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} = \sum_m f_m e^{\mathrm{i}km} \tag{9}$$

$$\mathcal{H}_{\text{Full s-wave}}(k) = \begin{pmatrix} -\mu - 2v\cos(k) & 0 & 0 & -\Delta \\ 0 & -\mu - 2v\cos(k) & \Delta & 0 \\ 0 & \Delta^* & \mu + 2v\cos(k) & 0 \\ -\Delta^* & 0 & 0 & \mu + 2v\cos(k) \end{pmatrix}. \tag{10}$$

Due to the absence of spin mixing terms and p-wave like pair correlations, this model can be separated in-to two disjoint copies. The physics is similar in both hence we shall focus on the "outer" block and were it is necessary we shall discuss relevant changes for the "inner" block. For simplicity the inner block will be referred to as $\mathcal{H}_{\text{s-wave}}(k)$, with elements

$$\mathcal{H}_{\text{s-wave}}(k) = \begin{pmatrix} -\mu - 2v\cos(k) & -\Delta \\ -\Delta^* & \mu + 2v\cos(k) \end{pmatrix}. \tag{11}$$

Interpreting the spectrum of $\mathcal{H}_{\mathbf{BdG}}$ in the absence of superconductivity

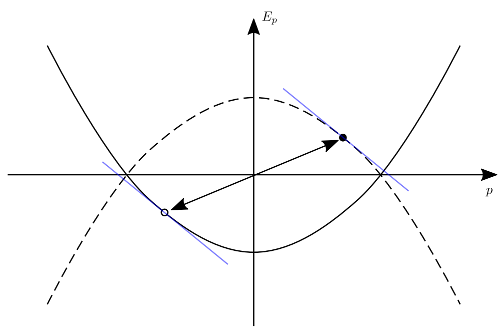

Figure 1: The spectrum of $\mathcal{H}_{\text{BdG}}$ in the absence of Δ. Particle like excitations are denoted by solid lines and hole like excitations are depicted by a dashed line. Taking out a particle below the Fermi sea corresponds to introducing a hole at positive energy. The two blue lines denote the parallel tangents corresponding to the group velocities of particle and hole excitations above and below the Fermi level as a graphical proof that these particles have the same velocity.

- The positive-energy half of the BdG spectrum contains all the physical information: the eigenstates can be used to construct the single-particle excitations of the many-body Hamiltonian, and the eigenvalues give the corresponding excitation energies. The negative-energy part of the BdG spectrum is hence redundant. (We need to pay special attention to 0-energy eigenstates, as we discuss later).

- At positive energies for $\Delta = 0$ we have two disjoint part of the spectrum, particles and holes. (Discuss what we mean by particle and hole type excitations!!)

  We shall call an excitation a particle if its Nambu spinor is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, while we shall call an excitation a hole if its spinor is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In Fig. 1 we denote particles and holes with solid and dashed lines respectively. Draw attention to possible alternative definitions used in the literature.

- If a particle-type positive-eigenvalue BdG excitation with momentum k and energy E_k is present in the many-body state, then this many-body state has an excess energy E_k, and this many-body state carries an excess current $e\partial_k E_k$ compared to the ground state.

- A hole at momentum k and energy E_k corresponds to the **absence** of a particle from the Fermi sea at momentum $-k$ and energy $-E_k$. As such it carries **$-e$ charge** but crucially its **velocity is the same** as the particle whose absence it signifies, that is $\partial_k E_k$. (**HF:** show if this is true from simple Fock space 5 site TB ring with $\mu = 0, v < 0$!)

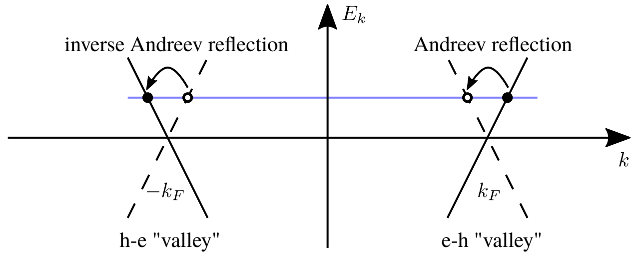

Figure 2: Linearized spectrum and Andreev processes.

Envelope function approximation for $k \approx \pm k_F$

Series expansion at k_F, here $-\mu - 2v\cos(k_F) = 0$

$$\mathcal{H}_{\text{Kitaev}}(k) \approx v_F \sigma_z (k - k_F) + 2\sin(k_F)\,\Delta\sigma_y \tag{12}$$

In the spirit of the EFA we relabel $k - k_F$ as -i$\partial_x = \hat{p}$ thus we arrive at:

$$\mathcal{H}^{eh}_{\text{Kitaev}} = \begin{pmatrix} v_F\hat{p} & -\mathrm{i}\tilde{\Delta} \\ \mathrm{i}\tilde{\Delta}^* & -v_F\hat{p} \end{pmatrix}, \tag{13}$$

where we introduced $v_F = |2v\sin(k_F)|$ and $\tilde{\Delta} = 2\sin(k_F)\,\Delta$. Note that for $\Delta = 0$, this EFA Hamiltonian describes particles propagating to the right and holes propagating to the left.

Series expansion at $-k_F$, gives the slightly different result:

$$\mathcal{H}^{he}_{\text{Kitaev}} = \begin{pmatrix} -v_F\hat{p} & \mathrm{i}\tilde{\Delta} \\ -\mathrm{i}\tilde{\Delta}^* & v_F\hat{p} \end{pmatrix} \tag{14}$$

notice that the sign of $\tilde{\Delta}$ changes! Note that for $\tilde{\Delta} = 0$, this EFA Hamiltonian describes particles propagating to the left and holes propagating to the right. Since the phase of $\tilde{\Delta}$ is not relevant for investigating junctions where only a single superconductor is present, we fix it such that i$\tilde{\Delta}$ is real and introduce $\Delta' = \mathrm{i}\tilde{\Delta}$. For the Kitaev wire hence the two valleys will be described by

$$\mathcal{H}^{eh}_{\text{Kitaev}} = \begin{pmatrix} v_F\hat{p} & -\Delta' \\ -\Delta' & -v_F\hat{p} \end{pmatrix}, \quad \mathcal{H}^{he}_{\text{Kitaev}} = \begin{pmatrix} -v_F\hat{p} & \Delta' \\ \Delta' & v_F\hat{p} \end{pmatrix} \tag{15}$$

For the s-wave the two valleys give the same sign in both valleys:

$$\mathcal{H}^{eh}_{\text{s-wave}} = \begin{pmatrix} v_F\hat{p} & -\Delta \\ -\Delta & -v_F\hat{p} \end{pmatrix}, \quad \mathcal{H}^{he}_{\text{s-wave}} = \begin{pmatrix} -v_F\hat{p} & -\Delta \\ -\Delta & v_F\hat{p} \end{pmatrix} \tag{16}$$

and we fix the phase of Δ to make it real.

We shall use the above derived EFA matrices in the spirit of the EFA approach and make the Δ-s position dependent. We shall be first and foremost interested in interfaces between a conducting channel and a superconductor. In this case we need to take in to account both of the "valley"-s at which we just produced

Landauer's approach extended for superconductors

Recap Landauer's approach for a conventional 1D single modded wire.

Recall from the previous semester that according to Landauer the current at finite bias of a wire with a single mode (no spin degeneracy assumed) in the presence of a scatterer is

$$I = \frac{e^2}{h}\int \underbrace{T(E)}_{1-R(E)}\,[f_L(E) - f_R(E)]\,\mathrm{d}E. \tag{17}$$

Where $T(E)$ ans $R(E)$ is the transmission and reflection probability of a particle impinging on the scatterer, $f_L(E)$and $f_R(E)$ are Fermi-Dirac distribution functions of particle reservoirs, possibly at different chemical potential. The simple relation $T = 1 - R$ is a consequence of the continuity relation of the particle flow expressed by the unitarity of the scattering matrix.

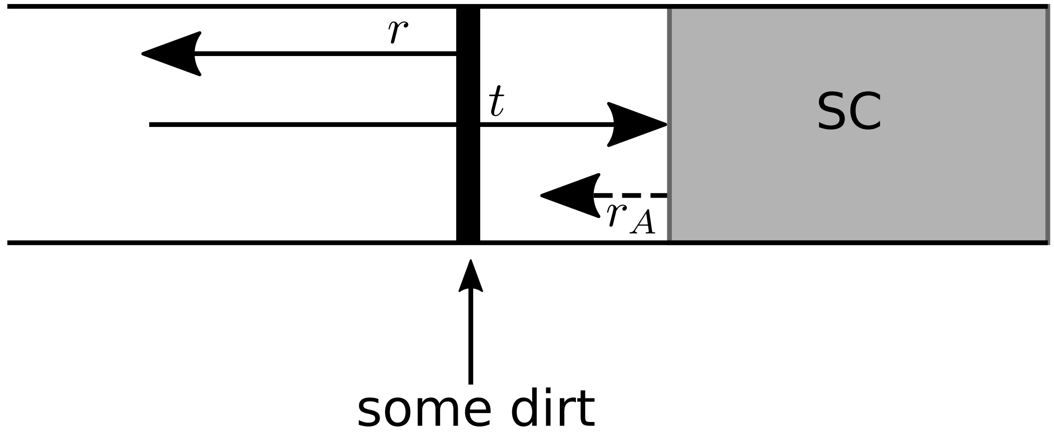

Figure 3: Elementary processes for an impinging particle at a dirty SN junction. A particle can either be reflected from the dirt with amplitude r or transmitted through it with amplitude t. From the superconductor it can be reflected back as a hole with amplitude r_A. As a consequence of these elementary processes there is a finite probability of reflection as a particle R and a finite probability as a reflection as a hole A.

Andreev reflection

As we saw above in the BdG picture we have to take in to account both particles and holes at a given energy. Further more holes have to be counted as carriers with opposite charge as particles!

In the case of a superconducting electrode particles can be converted to holes thus we have an additional term $A(E)$ describing Andreev scattering the process by which a superconductor converts particle like excitations impinging on it to hole like excitations and vice versa .

$$I = \frac{e^2}{h} \int (1 - R(E) + A(E)) \left[f_L(E) - f_R(E)\right] \mathrm{d}E \tag{18}$$

At zero temperature, assuming a finite bias voltage V such that $f_L(E) = f_R(E - eV)$, *i.e.* we are forcing particles from a reservoir towards the superconductor we have

$$I = \frac{e^2}{h} \int_0^{eV} (1 - R(E) + A(E)) \, \mathrm{d}E \tag{19}$$

thus the differential conductance at finite bias is

$$\frac{\mathrm{d}I}{\mathrm{d}V} = \frac{e^2}{h} (1 - R(eV) + A(eV)) \tag{20}$$

For energies below the gap R and A are related by unitarity of the scattering process $R + A = 1$, thus for bias voltages smaller than the gap we have

$$\frac{\mathrm{d}I}{\mathrm{d}V} = \frac{2e^2}{h} A(eV) = \left| r_A^{\mathrm{dirty}}(eV) \right|. \tag{21}$$

In what follows we restrict ourselves to this regime that is $E < \Delta$

Clean interfaces

If the interface between the normal and superconducting region is clean, than the only process which is allowed is Andreev scattering. In order to find the Andreev r_A and inverse Andreev $\tilde{r}_A$ scattering coefficients we resort to mode matching of scattering wavefunctions. (See notebook and the appendix!)

This procedure yields

$$r_A = \frac{-E + \mathrm{i}\sqrt{\Delta^2 - E^2}}{\Delta} = \mathrm{e}^{\mathrm{i} \arccos\left(-\frac{E}{\Delta}\right)}, \tag{22}$$

$$\tilde{r}_A = \frac{\pm\Delta}{E + \mathrm{i}\sqrt{\Delta^2 - E^2}} = \pm \mathrm{e}^{-\mathrm{i} \arccos\left(\frac{E}{\Delta}\right)} \tag{23}$$

where the + sign is for the p-wave case and the − sign is for the s-wave.

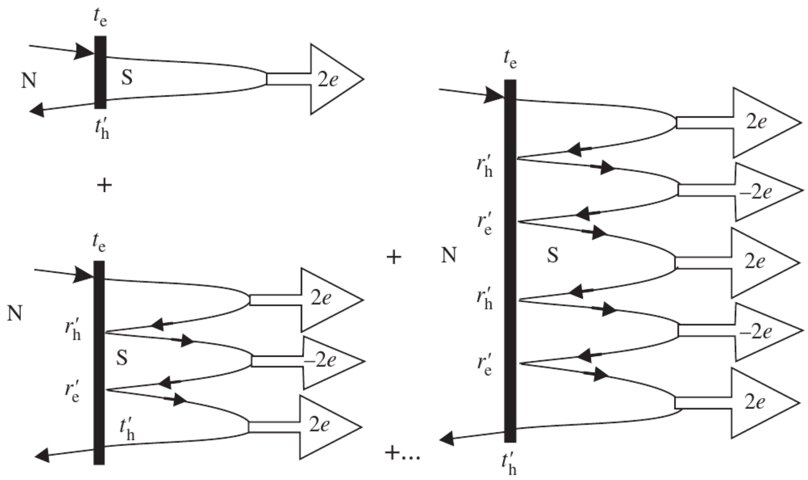

Figure 4: Elementary processes for Andreev reflection of a particle from a dirty NINS interface

Scattering processes at a generic SN interface

Consider a SN interface with some scattering potential in front of the superconductors To find the total amplitude of Andreev scattering we sum up all paths where a hole is reflected back to the normal contact.

$$r_A^{\text{dirty}} = t'_h \left[1 + (r_A r'_e \tilde{r}_A r'_h) + ()^2 + \dots\right] r_A t_e = \frac{t'_h r_A t_e}{1 - r_A r'_e \tilde{r}_A r'_h} \tag{24}$$

The magnitude of a_{he} gives the total probability of Andreev reflection

$$A(E) = \left|r_A^{\text{dirty}}(E)\right|^2. \tag{25}$$

In an extremely simple yet generic enough approach we can assume that $t_e = t'_h = t$ similarly $r'_e = r'_h = r$ and that they do not depend on energy and are related by unitarity $t^2 + r^2 = 1$,

$$A(E) = \left|\frac{r_A t^2}{1 - (1 - t^2)\, r_A \tilde{r}_A}\right|^2 \tag{26}$$

An important consequence of the additional $-$ sign for the inverse Andreev reflection process in the s-wave case can deduced for $E = 0$:

$$A_{\text{Kitaev}}(0) = |r_A|^2 = 1 \tag{27}$$

and

$$A_{\text{s-wave}}(0) = \left|\frac{r_A t^2}{2 - t^2}\right|^2. \tag{28}$$

That is for a topological **p-wave** superconductor **at zero bias dI/dV** is independent for the dirt and **is quantized to $\frac{2e^2}{h}$**! For **s-wave** the probability of Andreev reflection will still **depend on the details** of the dirty interface, in this case on t. If $t = 1$, $dI/dV = 2 \times \frac{2e^2}{h}$ but otherwise it can take on any value!

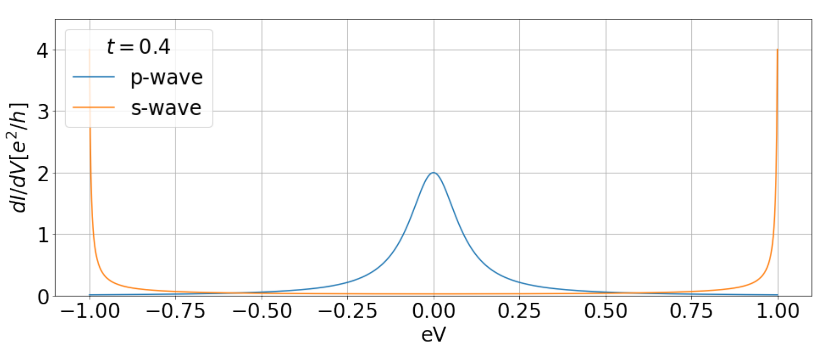

Figure 5: dI/dV for p-wave and s-wave NINS junctions for $\Delta = 1$.

Appendix

Mode matching

Consider the elementary process where by in the Kitaev model a particle is converted to a hole. for this we investigate the EFA Hamiltonian in the e-h valley:

$$\mathcal{H}^{eh}_{\text{Kitaev}} = \begin{pmatrix} v_F\hat{p} & -\Delta' \\ -\Delta' & -v_F\hat{p} \end{pmatrix}. \tag{29}$$

On the left side of the junction at $x < 0$ we set $\Delta' = 0$ and thus a scattering wavefunction describing particles impinging to the interface have the form

$$\psi^{eh}|_{x<0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}kx} + \begin{pmatrix} 0 \\ r_A \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx}. \tag{30}$$

Inside the superconductor there are no propagating states below Δ', but evanescent solutions for the EFA exist. The

$$\psi^{eh}|_{x>0} = \begin{pmatrix} u \\ v \end{pmatrix} \mathrm{e}^{-\kappa x} \tag{31}$$

Substituting this ansatz into $\mathcal{H}^{eh}_{\text{Kitaev}}\psi = E\psi$ yields u, v and κ. Matching the solutions at $x = 0$, we have

$$\begin{pmatrix} 1 \\ r_A \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \rightarrow r_A = \frac{v}{u} = \frac{-E + \mathrm{i}\sqrt{\Delta^2 - E^2}}{\Delta}. \tag{32}$$

Note that this will be the same fort the s-wave case!

For the inverse process we need to use $\mathcal{H}^{he}_{\text{Kitaev}}$ and $\mathcal{H}^{he}_{\text{s-wave}}$ which crucially differ by the sign of their respective Δ! The scattering wavefunction on the normal side now is:

$$\psi^{he}|_{x<0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}kx} + \begin{pmatrix} \tilde{r}_A \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx}. \tag{33}$$

In the superconductor we again take a decaying ansatz

$$\psi^{he}|_{x>0} = \begin{pmatrix} u' \\ v' \end{pmatrix} \mathrm{e}^{-\kappa x} \tag{34}$$

Matching the solutions at $x = 0$, we have

$$\begin{pmatrix} \tilde{r}_A \\ 1 \end{pmatrix} = \begin{pmatrix} u' \\ v' \end{pmatrix}, \rightarrow \tilde{r}_A = \frac{u'}{v'} = \frac{\pm\Delta}{E + \mathrm{i}\sqrt{\Delta^2 - E^2}}. \tag{35}$$

where the $+$ sign is for the p-wave case and the $-$ sign is for the s-wave.