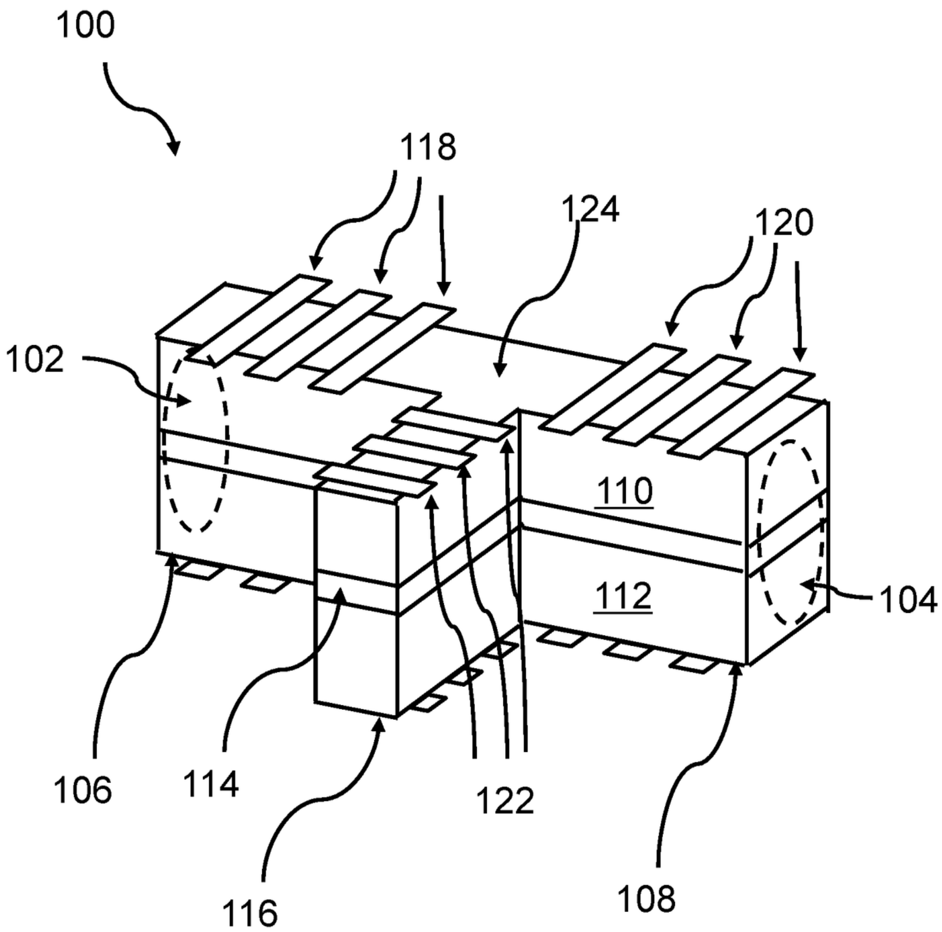
Parafermions

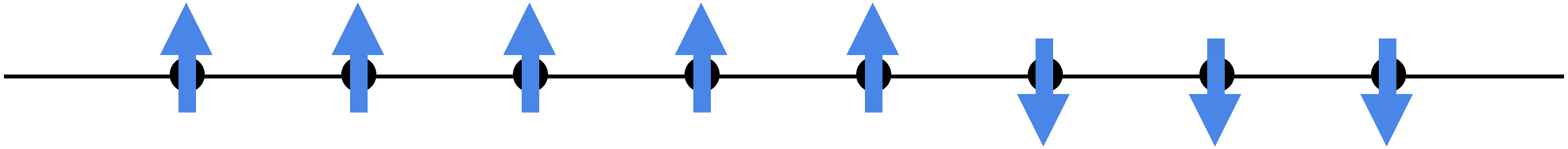


Parafermion braiding device US patent No. 10 297 739B1

Outline

- Ising model and Kitaev model
- Clock models and parafermions
- Parafermions from ordinary fermions

Ising model



$$H = -J \sum_{j=1}^{L-1} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z - f \sum_j^L \hat{\sigma}_j^x$$

$$\left(\hat{\sigma}_j^z\right)^2 = \left(\hat{\sigma}_j^x\right)^2 = \hat{1}, \quad \hat{\sigma}_j^x \hat{\sigma}_j^z = -\hat{\sigma}_j^z \hat{\sigma}_j^x, \quad \hat{\sigma}_q^x \hat{\sigma}_p^z = \hat{\sigma}_p^z \hat{\sigma}_q^x \ (p \neq q)$$

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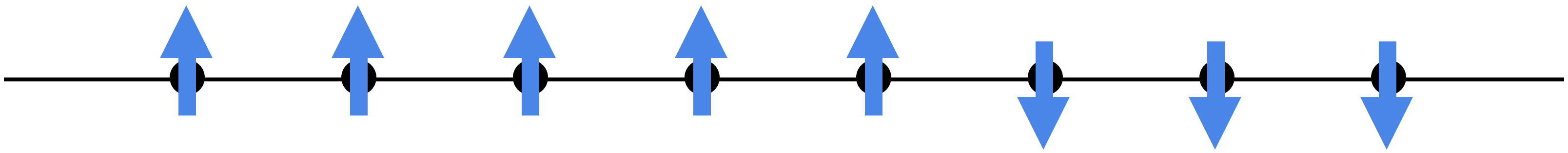

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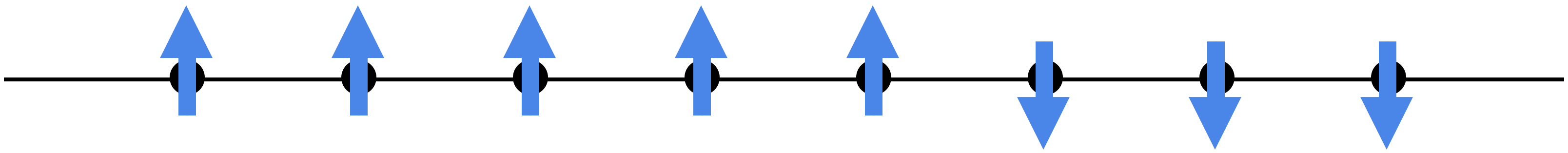
$$|\psi\rangle = \sum_{\sigma_i=\pm 1} \psi_{\sigma_1\sigma_2\ldots\sigma_L}\,|\sigma_1\sigma_2\ldots\sigma_L\rangle \qquad |1,-1,1,-1,1\rangle \equiv |{\uparrow\downarrow\uparrow\downarrow\uparrow}\rangle$$

Ising model



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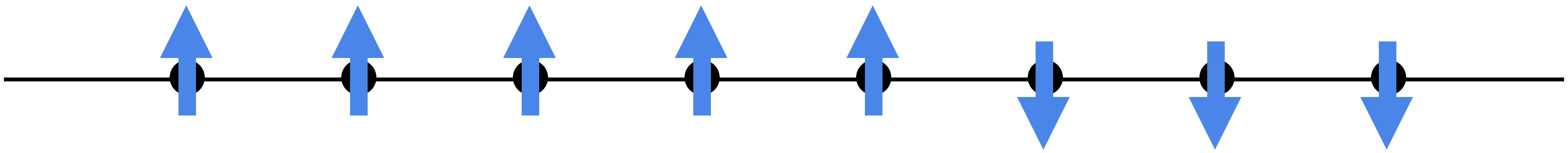
$$\hat{\sigma}_j^z\,|\sigma_1\sigma_2\ldots\sigma_j\ldots\sigma_L\rangle = \sigma_j\,|\sigma_1\sigma_2\ldots\sigma_j\ldots\sigma_L\rangle$$

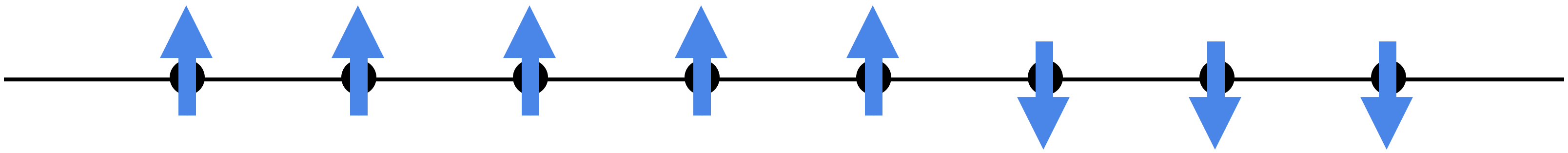
$$\hat{\sigma}_j^x\,|\sigma_1\sigma_2\ldots\sigma_j\ldots\sigma_L\rangle = |\sigma_1\sigma_2\ldots\bar{\sigma}_j\ldots\sigma_L\rangle$$

Ising model



$$H = -J \sum_{j=1}^{L-1} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z - f \sum_j^L \hat{\sigma}_j^x$$

Ising model



$$H = -J\sum_{j=1}^{L-1}\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z - f\sum_j^L \hat{\sigma}_j^x$$

$$f = 0$$

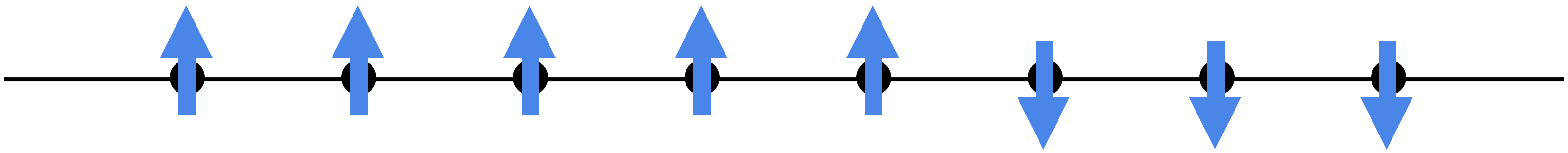
Ground state doubly degenerate:

$$|GS\rangle_1 = |\uparrow\uparrow\cdots\uparrow\rangle$$

$$|GS\rangle_2 = |\downarrow\downarrow\cdots\downarrow\rangle$$

$$\langle\hat{\sigma}_j^z\rangle \neq 0$$

Ising model



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$$\langle\hat{\sigma}^z_j\rangle \neq 0$$

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Ground state nondegenerate:

$$|GS\rangle = \bigotimes_j \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$

$$\langle\hat{\sigma}^z_j\rangle = 0$$

Jordan-Wigner transformation

(Fradkin-Kadanoff)

$$\hat{\gamma}_{2p-1} = \hat{\sigma}_p^z \prod_{q<p} \hat{\sigma}_q^x$$

$$\hat{\gamma}_{2p} = \mathrm{i}\hat{\sigma}_p^x \hat{\sigma}_p^z \prod_{q<p} \hat{\sigma}_q^x$$

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$$\hat{\gamma}_k^2 = \hat{1}, \hat{\gamma}_k^\dagger = \hat{\gamma}_k$$

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Chi l'ha visto?

Ettore Majorana, ordinario di fisica teorica all' Università di Napoli, è misteriosamente scomparso dagli ultimi di marzo. Di anni 31, alto metri 1,70, snello, con capelli neri, occhi scuri, una lunga cicatrice sul dorso di una mano. Chi ne sapesse qualcosa è pregato di scrivere al R. P. E. Marianecci, Viale Regina Margherita 66 - Roma.

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$$\hat{\gamma}^2_k = \hat{1}, \hat{\gamma}^\dagger_k = \hat{\gamma}_k$$

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This is a mapping from $\frac{1}{2}$ spin to Majorana operators !

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Hamiltonian in terms of Majoranas

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$$\Updownarrow$$

$$H = -J \sum_{p=1}^{L-1} \mathrm{i}\hat{\gamma}_{2p}\hat{\gamma}_{2p+1} - f \sum_{p=1}^{L-1} \mathrm{i}\hat{\gamma}_{2p-1}\hat{\gamma}_{2p}$$

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$$\Updownarrow$$

$$H = -J \sum_{p=1}^{L-1} \left(\hat{a}_p^\dagger \hat{a}_{p+1}^\dagger + \hat{a}_{p+1}^\dagger \hat{a}_p + \text{h. c.} \right) - f \sum_{p=1}^{L-1} \left(-2 \hat{a}_p^\dagger \hat{a}_p + \hat{1} \right)$$

$$\hat{\gamma}_{2p-1} = \hat{a}_p^\dagger + \hat{a}_p, \quad \hat{\gamma}_{2p} = \frac{\hat{a}_p^\dagger - \hat{a}_p}{\mathbf{i}}, \quad \hat{a}_p^\dagger \hat{a}_q + \hat{a}_q \hat{a}_p^\dagger = \delta_{pq}$$

Hamiltonian in terms of Majoranas

$$H = -J \sum_{p=1}^{L-1} \hat{\sigma}_p^z \hat{\sigma}_{p+1}^z - f \sum_{p=1}^{L-1} \hat{\sigma}_p^x$$

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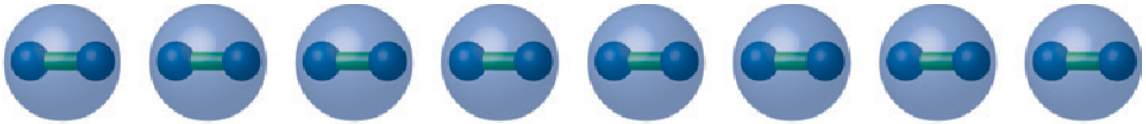
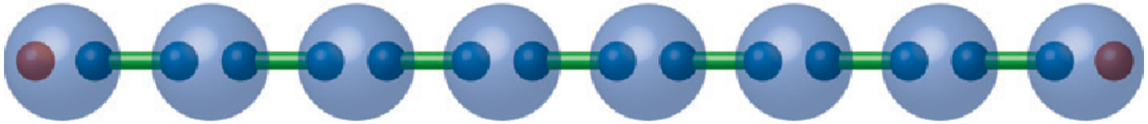
$$H = -J \sum_{p=1}^{L-1} \mathrm{i}\hat{\gamma}_{2p}\hat{\gamma}_{2p+1} - f \sum_{p=1}^{L-1} \mathrm{i}\hat{\gamma}_{2p-1}\hat{\gamma}_{2p}$$

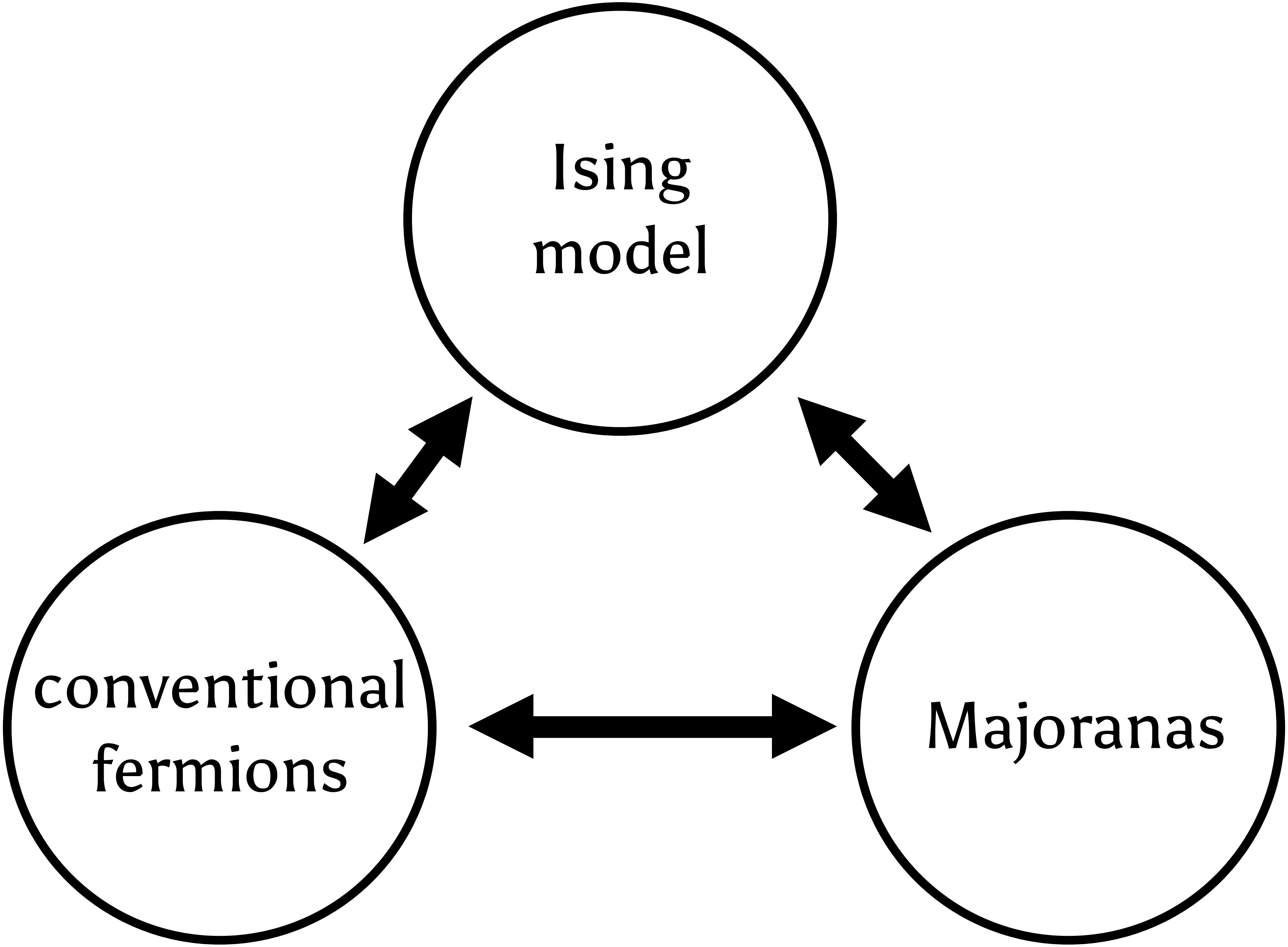
$$\Updownarrow$$

$J = 0$  unique groundstate

$f = 0$  degenerate groundstate

This is the Kitaev wire!

even/odd parity


Ising
model
conventional
fermions
Majoranas

Generalizing the Ising model

N degrees of freedom on each site:

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$$N = 2 : \quad |\uparrow\rangle, |\downarrow\rangle$$ Ising model

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Generalizing the Ising model

N degrees of freedom on each site:

$$N = 2: \quad |\uparrow\rangle, |\downarrow\rangle$$ Ising model

$$N = 3: \quad |\uparrow\rangle, |\swarrow\rangle, |\searrow\rangle$$

$$N = 4: \quad |\uparrow\rangle, |\leftarrow\rangle, |\downarrow\rangle, |\rightarrow\rangle$$

...

Single site clock models

$$N = 3: \quad |\uparrow\rangle, |\swarrow\rangle, |\searrow\rangle$$

Single site clock models

$$N = 3 : \quad |\uparrow\rangle, |\swarrow\rangle, |\searrow\rangle$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \omega = \mathrm{e}^{\mathrm{i}2\pi/N}$$

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"reading time" and "winding the clock"

$\sigma|\uparrow\rangle = 1|\uparrow\rangle$	$\tau|\uparrow\rangle = |\swarrow\rangle$
$\sigma|\swarrow\rangle = \omega|\swarrow\rangle$	$\tau|\swarrow\rangle = |\searrow\rangle$
$\sigma|\searrow\rangle = \omega^2|\searrow\rangle$	$\tau|\searrow\rangle = |\uparrow\rangle$

Single site clock models

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$$\sigma|\uparrow\rangle = 1|\uparrow\rangle$$
$$\sigma|\swarrow\rangle = \omega|\swarrow\rangle$$
$$\sigma|\searrow\rangle = \omega^2|\searrow\rangle$$

$$\tau|\uparrow\rangle = |\swarrow\rangle$$
$$\tau|\swarrow\rangle = |\searrow\rangle$$
$$\tau|\searrow\rangle = |\uparrow\rangle$$

$$\tau^N = \sigma^N = 1$$
$$\sigma^\dagger = \sigma^{N-1}$$
$$\tau^\dagger = \tau^{N-1}$$
$$\sigma\tau = \omega\tau\sigma$$

Note that σ and τ are not Hermitian!

1D chain clock models

$$H = -J\mathrm{e}^{\mathrm{i}\phi}\sum_{p=1}^{L-1}\hat{\sigma}_p^\dagger\hat{\sigma}_{p+1} - f\mathrm{e}^{\mathrm{i}\theta}\sum_{p=1}^{L-1}\hat{\tau}_p + \mathrm{h.c.}$$

$$\hat{\sigma}_p = \left(\bigotimes_{q=1}^{p-1}\hat{1}\right)\otimes\sigma\otimes\left(\bigotimes_{q=p+1}^{L}\hat{1}\right)$$

$$\hat{\tau}_p = \left(\bigotimes_{q=1}^{p-1}\hat{1}\right)\otimes\tau\otimes\left(\bigotimes_{q=p+1}^{L}\hat{1}\right)$$

$$\hat{\sigma}_p^N = \hat{\tau}_p^N = \hat{1}$$

$$\hat{\sigma}_p^{N-1} = \hat{\sigma}_p^\dagger,\, \hat{\tau}_p^{N-1} = \hat{\tau}_p^\dagger$$

$$\hat{\sigma}_p\hat{\tau}_p = \omega\hat{\tau}_p\hat{\sigma}_p$$

$$\hat{\sigma}_p\hat{\tau}_q = \hat{\tau}_q\hat{\sigma}_p,\, p \neq q$$

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$$f = 0$$

Ground state N-fold degenerate

$$N = 3$$
$$|GS\rangle_1 = |\uparrow\uparrow \cdots \uparrow\rangle$$
$$|GS\rangle_2 = |\swarrow\swarrow \cdots \swarrow\rangle$$
$$|GS\rangle_3 = |\searrow\searrow \cdots \searrow\rangle$$

$$\langle \hat{\sigma}_p \rangle \neq 0$$

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$$\langle\hat{\sigma}_p\rangle \neq 0$$

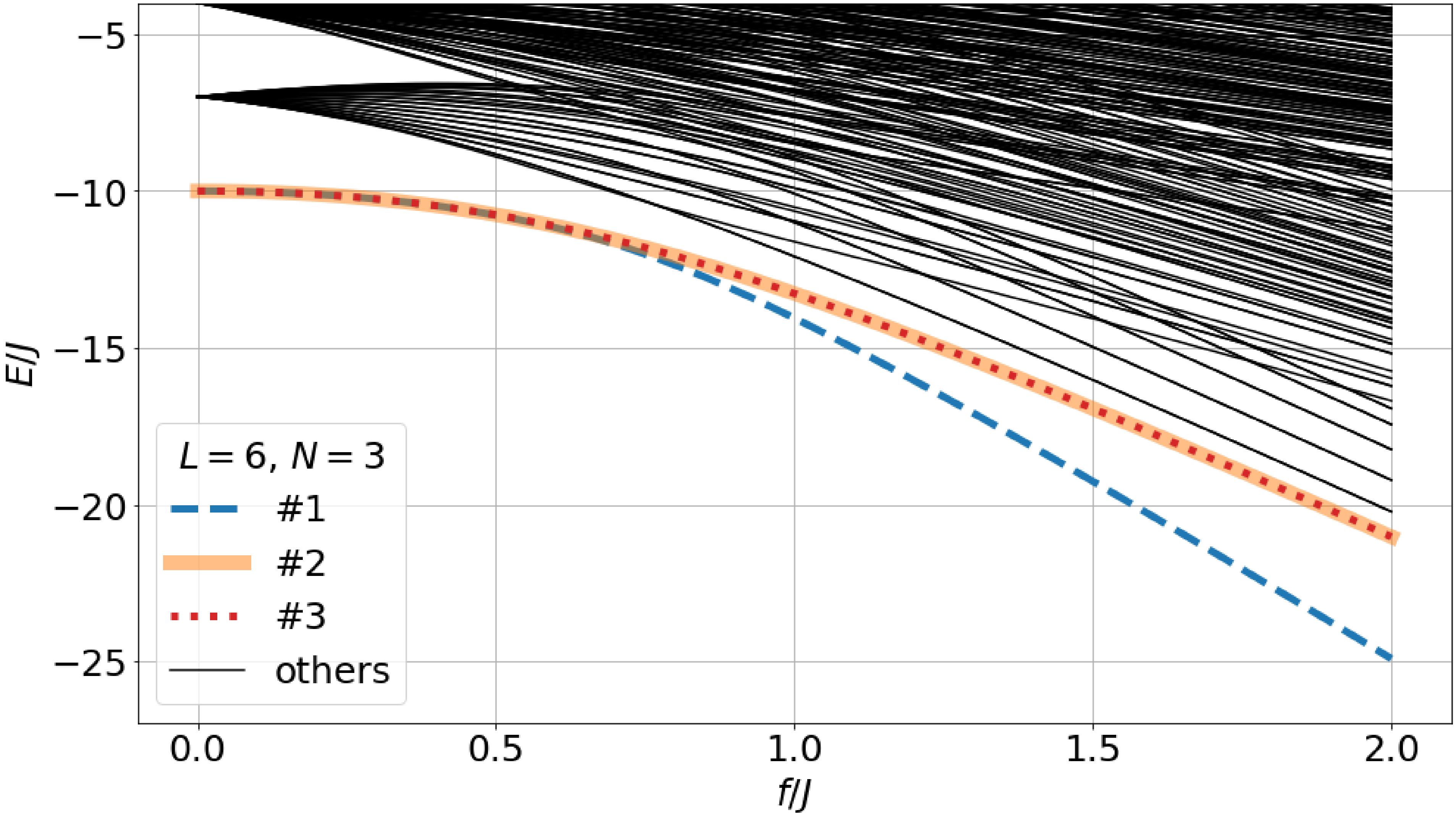
$J = 0$

Ground state nondegenerate

$$|GS\rangle = \bigotimes_j \frac{|\uparrow\rangle + |\searrow\rangle + |\swarrow\rangle}{\sqrt{3}}$$

$$\langle\hat{\sigma}_p\rangle = 0$$

Spectrum of a finite length N=3 clock model



Jordan-Wigner transformation

$$\hat{\alpha}_{2p-1} = \hat{\sigma}_p \prod_{q<p} \hat{\tau}_q$$

$$\hat{\alpha}_{2p} = \varpi \hat{\tau}_p \hat{\sigma}_p \prod_{q<p} \hat{\tau}_q$$

$$\varpi = -\mathrm{e}^{\mathrm{i}\pi/N}$$

$$\omega = \mathrm{e}^{\mathrm{i}2\pi/N}$$

Jordan-Wigner transformation

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The definition of **parafermions**!

for N=2 we get Majoranas!

Hamiltonian in terms of parafermions

$$H = -Je^{\mathrm{i}\phi} \sum_{p=1}^{L-1} \hat{\sigma}_p^\dagger \hat{\sigma}_{p+1} - f e^{\mathrm{i}\theta} \sum_{p=1}^{L-1} \hat{\tau}_p + \mathrm{h.c.}$$

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$$\Updownarrow$$

$$H = -J\mathrm{e}^{\mathrm{i}\phi} \sum_{p=1}^{L-1} \varpi \hat{\alpha}_{2p}^\dagger \hat{\alpha}_{2p+1} - f\mathrm{e}^{\mathrm{i}\theta} \sum_{p=1}^{L} \varpi \hat{\alpha}_{2p-1}^\dagger \hat{\alpha}_{2p}$$

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$$\Updownarrow$$

$$H = -J\mathrm{e}^{\mathrm{i}\phi}\sum_{p=1}^{L-1}\varpi\hat{\alpha}_{2p}^\dagger\hat{\alpha}_{2p+1} - f\mathrm{e}^{\mathrm{i}\theta}\sum_{p=1}^{L}\varpi\hat{\alpha}_{2p-1}^\dagger\hat{\alpha}_{2p}$$

$f = 0 \rightarrow$ edge localized parafermions, $\hat{\alpha}_1$ & $\hat{\alpha}_{2L}$, missing from the Hamiltonian!

In this phase we can **encode an N-fold degenerate subspace in** the missing **two parafermion modes**!

Zero modes and Symmetries

Strong Zero mode:

- operative definition: exist a symmetry represented by $\hat{\mathcal{Q}}$ and exist an operator $\hat{\alpha}$ such that $[\hat{H}, \hat{\alpha}] = 0 + \mathcal{O}(e^{-L})$ and $[\hat{\mathcal{Q}}, \hat{a}] \neq 0$
- consequence: ground state **AND** excited states are degenerate

Weak Zero mode:

- only ground state is degenearte

Zero modes and Symmetries

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- operative definition: exist a symmetry represented by $\hat{\mathcal{Q}}$ and exist an operator $\hat{\alpha}$ such that $[\hat{H}, \hat{\alpha}] = 0 + \mathcal{O}(e^{-L})$ and $[\hat{\mathcal{Q}}, \hat{a}] \neq 0$
- consequence: ground state **<u>AND</u>** excited states are degenerate

<u>Weak Zero mode:</u>

- only ground state is degenearte

<u>Parafermion-clock models</u>

$\exists$ symmetry !

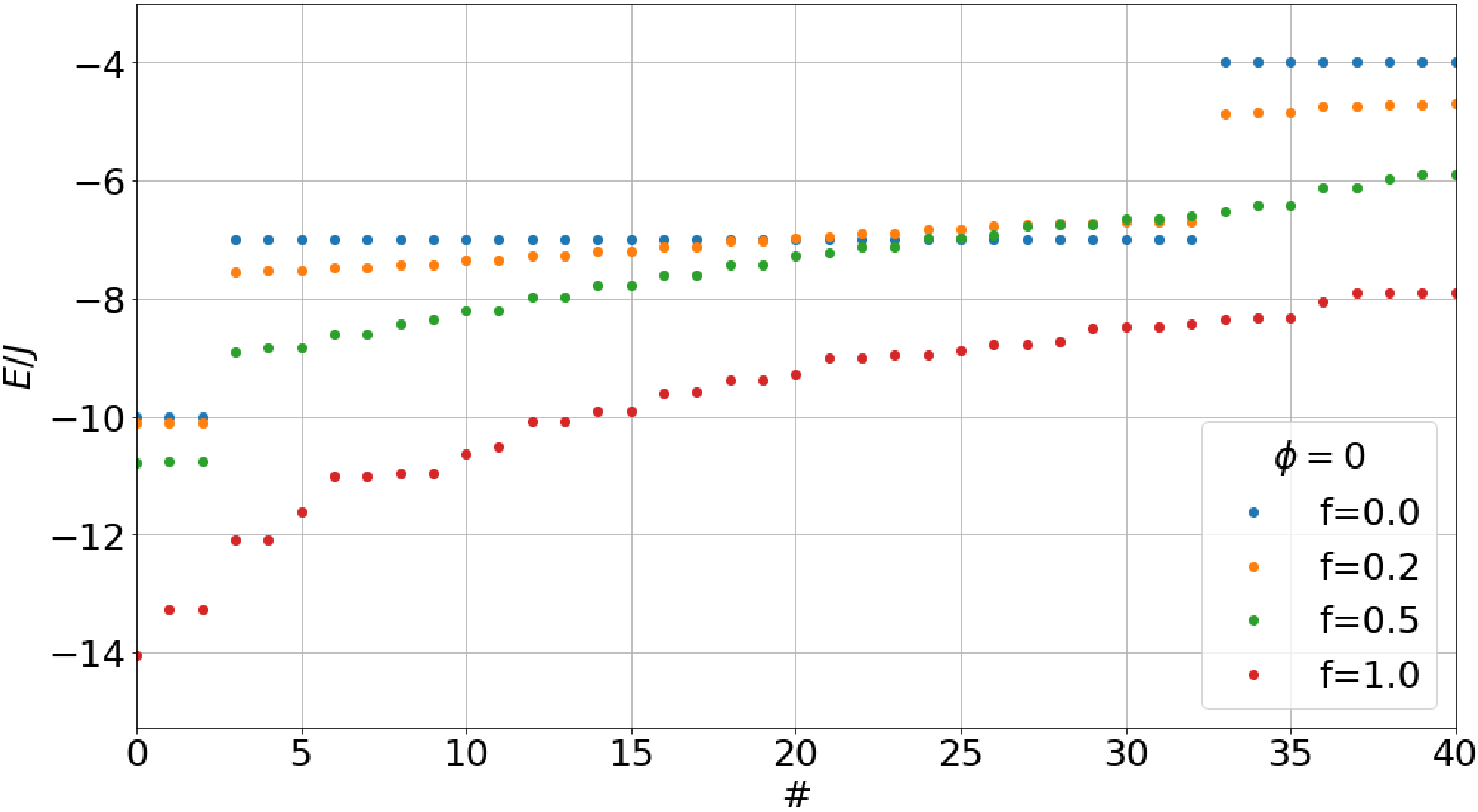
N-ality, parafermion "parity"

$$\hat{\mathcal{Q}} = \prod_p \hat{\tau}_p$$

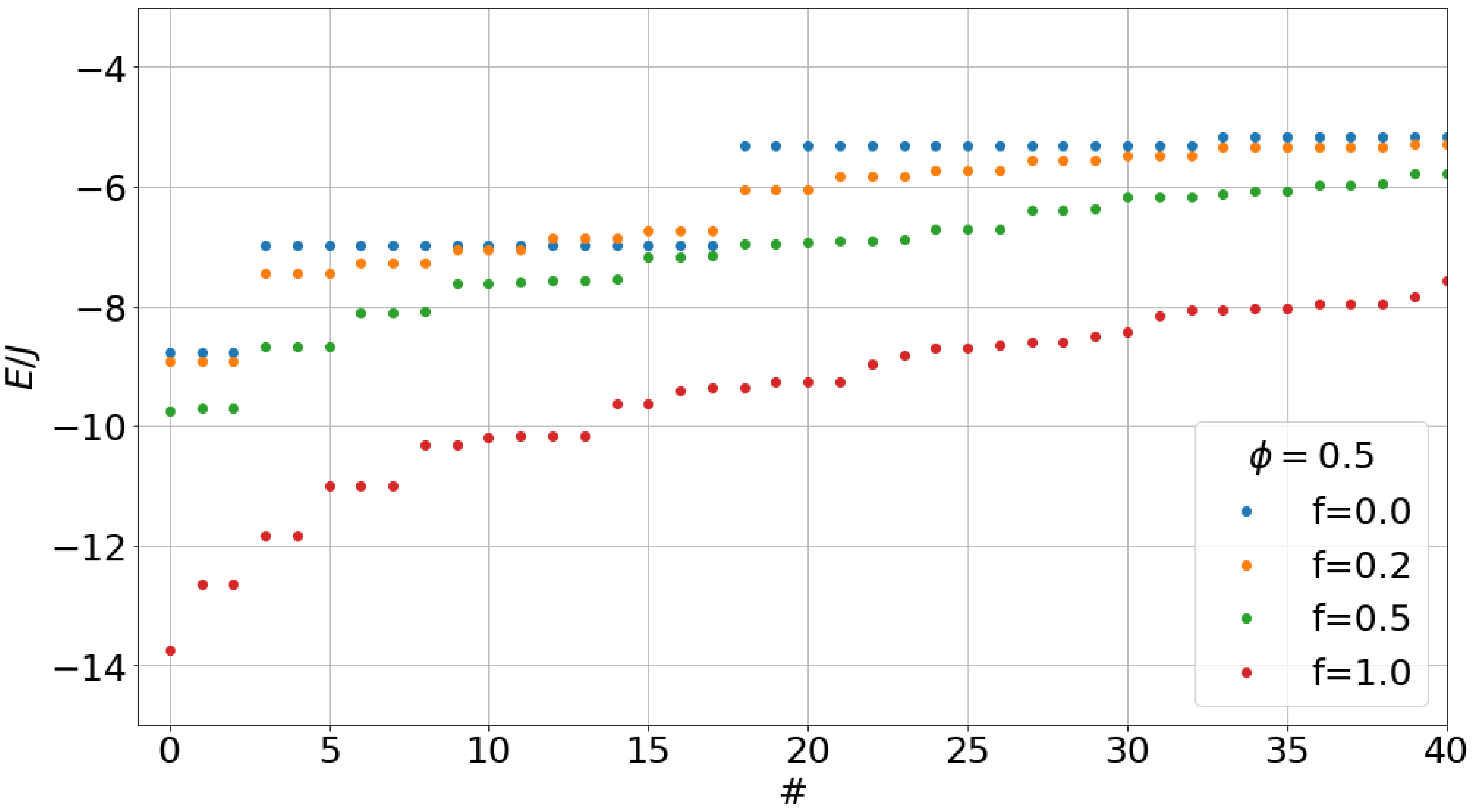
$\hat{\alpha}_1$ and $\hat{\alpha}_{2L}$ commute with $\hat{H}$ for $f = 0$!

$$H = -J\mathrm{e}^{\mathrm{i}\phi} \sum_{p=1}^{L-1} \hat{\sigma}_p^\dagger \hat{\sigma}_{p+1} - f\mathrm{e}^{\mathrm{i}\theta} \sum_{p=1}^{L-1} \hat{\tau}_p + \mathrm{h.c.}$$

- $f = 0$: strong zero modes
- $f \neq 0, \phi = \theta = 0$: zero modes are weak
  c.f. perturbative picture
- $f \neq 0, \phi \neq 0$: ?



−4
−6
−8
−10
−12
−14
E/J
0
5
10
15
20
25
30
35
40
#
$\phi = 0$
f=0.0
f=0.2
f=0.5
f=1.0



E/J
−4
−6
−8
−10
−12
−14
0
5
10
15
20
25
30
35
40
#
$\phi = 0.5$
f=0.0
f=0.2
f=0.5
f=1.0

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Parafermionic clock models and quantum resonance

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We explore the $\mathbb{Z}_N$ parafermionic clock-model generalizations of the p-wave Majorana wire model. In particular, we examine whether zero-mode operators analogous to Majorana zero modes can be found in these models when one introduces chiral parameters to break time reversal symmetry. The existence of such zero modes implies N-fold degeneracies throughout the energy spectrum. We address the question directly through these degeneracies by characterizing the entire energy spectrum using perturbation theory and exact diagonalization. We find that when N is prime, and the length L of the wire is finite, the spectrum exhibits degeneracies up to a splitting that decays exponentially with system size, for generic values of the chiral parameters. However, at particular parameter values (resonance points), band crossings appear in the unperturbed spectrum that typically result in a splitting of the degeneracy at finite order. We find strong evidence that these preclude the existence of strong zero modes for generic values of the chiral parameters. In particular we show that in the thermodynamic limit, the resonance points become dense in the chiral parameter space. When N is not prime, the situation is qualitatively different, and degeneracies in the energy spectrum are split at finite order in perturbation theory for generic parameter values, even when the length of the wire L is finite. Exceptions to these general findings can occur at special "antiresonant" points. Here the evidence points to the existence of strong zero modes and, in the case of the achiral point of the $N = 4$ model, we are able to construct these modes exactly.

$\mathbb{Z}_4$ parafermions from ordinary fermions

A. Calzona, T. Meng, M. Sassetti, T. L. Schmidt Phys. Rev. B **98**, 201110(R) (2018)

$\mathbb{Z}_4$ parafermions from ordinary fermions

$\boxed{\text{N=4 clock model/ parafermion chain}}$

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$\mathbb{Z}_4$ parafermions from ordinary fermions

N=4 clock model/
parafermion chain

each site
has 4 states

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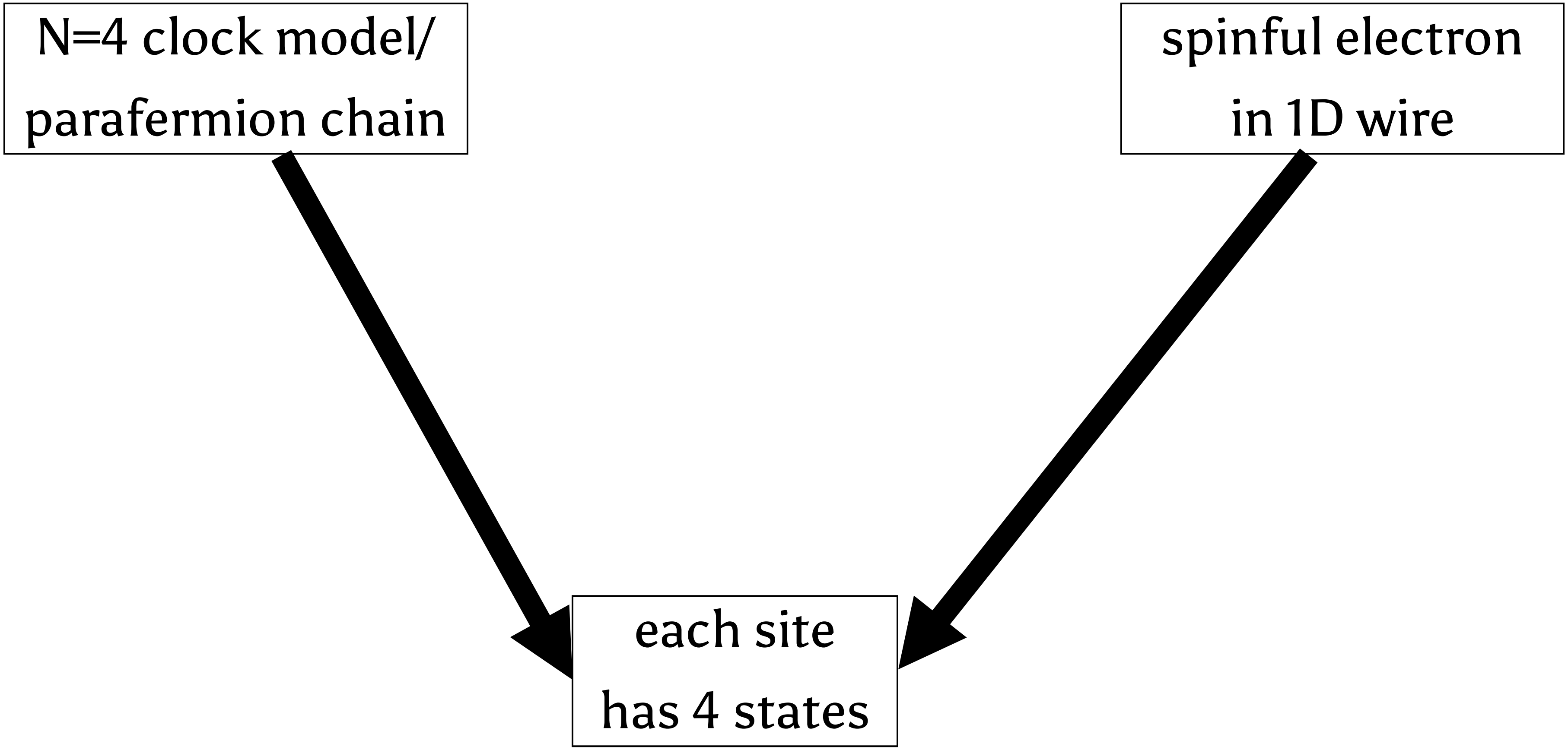

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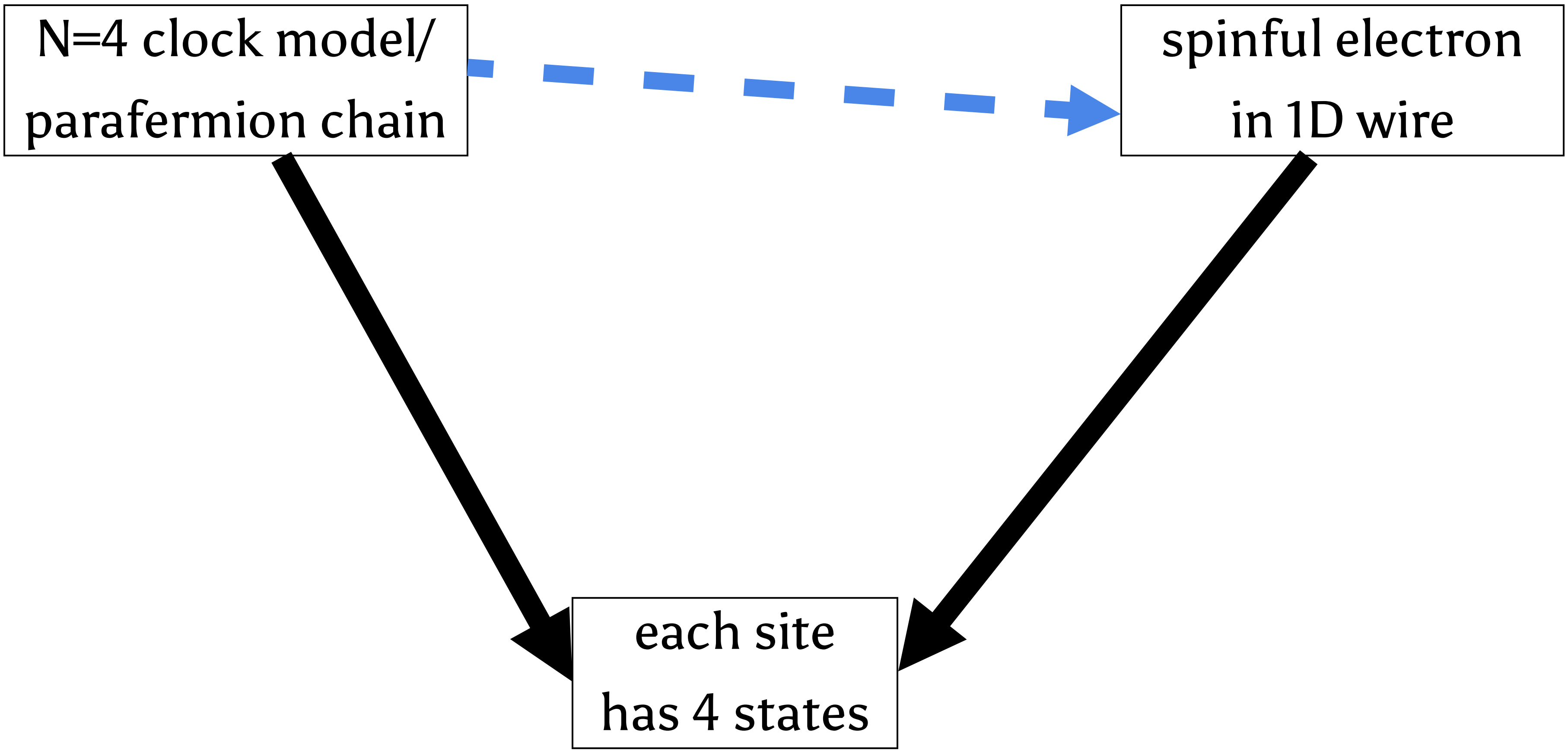

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$\mathbb{Z}_4$ parafermions from ordinary fermions

<u>steps of the derivation:</u>

$$H = -J\mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \sum_p b_p a_{p+1}^\dagger \qquad a_p = \alpha_{2p-1},\, b_p = \alpha_{2p}$$

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Introduce "Fock-parafermions"

$$a_j = d_j + d_j^{\dagger 3}, \quad b_j = e^{\mathrm{i}\pi/4}(d_j \mathrm{i}^{N_j} + d_j^{\dagger 3}), \quad N_j = \sum_{m=1}^{3} d_j^{\dagger m} d_j^m$$

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Identify local Fock space to that of spinful electrons

$$d_j = \mathrm{i}^{\sum_{p<j}(n_{p\downarrow}+3n_{p\uparrow}-2n_{p\uparrow}n_{p\downarrow})}\left(c_{j\uparrow} - c_{j\uparrow}n_{j\downarrow} - c^\dagger_{j\uparrow}n_{j\downarrow} + ic_{j\downarrow}n_{j\uparrow}\right)$$

$$n_{p\sigma} = c^\dagger_{p\sigma}c_{p\sigma}$$

$\mathbb{Z}_4$ parafermions from ordinary fermions

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Note: odd number of c-s in the expression
for d $\to H$ can be mapped to a local model

Hamiltonian in fermion language …

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$$
\begin{aligned}
H &= H^{(2)} + H^{(4)} + H^{(6)} \\
H^{(2)} &= -J \sum_{\sigma,j} \left[c^{\dagger}_{\sigma,j} c_{\sigma,j+1} - i\, c^{\dagger}_{-\sigma,j} c^{\dagger}_{\sigma,j+1} \right] + h.c.\,, \\
H^{(4)} &= -J \sum_{\sigma,j} \Big[c^{\dagger}_{\sigma,j} c_{\sigma,j+1} \left(-n_{-\sigma,j} - n_{-\sigma,j+1} \right) \\
&\quad + c^{\dagger}_{\sigma,j} c_{-\sigma,j+1}\; i \left(n_{-\sigma,j} + n_{\sigma,j+1} \right) \\
&\quad + c^{\dagger}_{-\sigma,j} c^{\dagger}_{\sigma,j+1}\; i \left(n_{\sigma,j} + n_{-\sigma,j+1} \right) \\
&\quad + c^{\dagger}_{\sigma,j} c^{\dagger}_{\sigma,j+1} \left(n_{-\sigma,j} - n_{-\sigma,j+1} \right) \Big] + h.c.\,, \\
H^{(6)} &= -J \sum_{j} \Big[-2i\, c^{\dagger}_{\sigma,j} c_{-\sigma,j+1} \left(n_{-\sigma,j} n_{\sigma,j+1} \right) \\
&\quad - 2i\, c^{\dagger}_{-\sigma,j} c^{\dagger}_{\sigma,j+1} \left(n_{\sigma,j} n_{-\sigma,j+1} \right) \Big] + h.c.
\end{aligned}
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Hamiltonian in fermion language …

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&\qquad - 2i\, c^\dagger_{-\sigma,j} c^\dagger_{\sigma,j+1} \left(n_{\sigma,j} n_{-\sigma,j+1} \right) \Big] + h.c.
\end{aligned}
$$

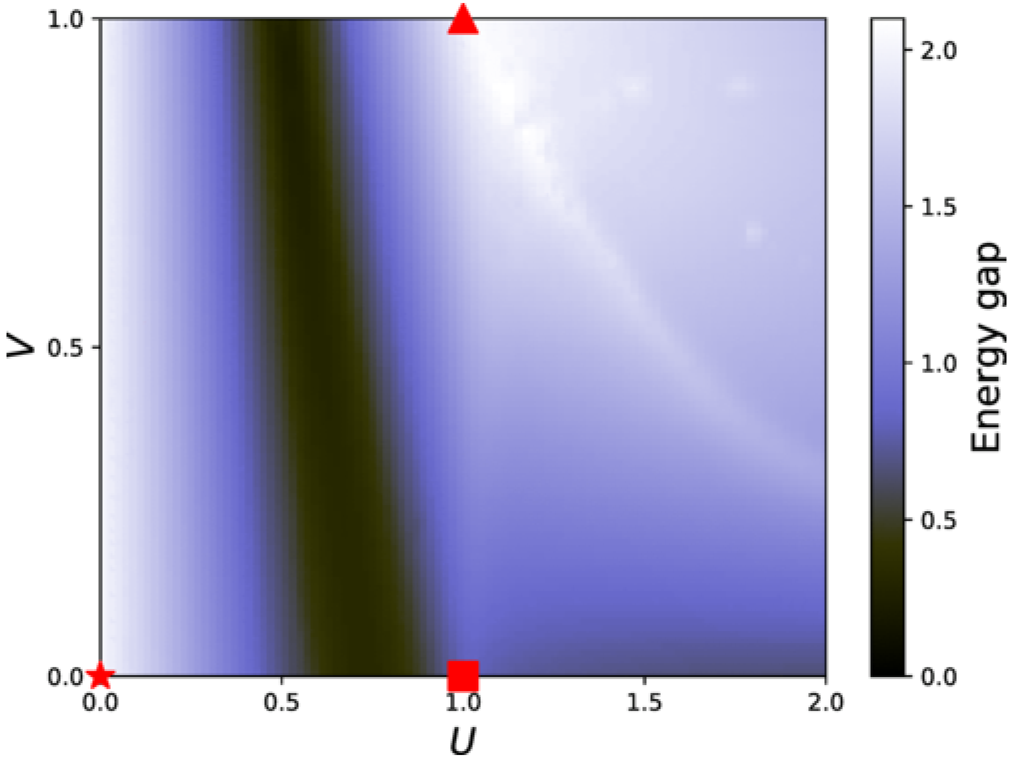
… is complicated with 3 body interactions encoded in the $H^{(6)}$ term

Interpolating to quartic Hamiltonians can preserve the gap!

$$\bar{H}(U,V) = H^{(2)} + U\left[V\left(H^{(4)} + H^{(6)}\right) + (1-V)\bar{H}^{(4)}\right]$$

$$\bar{H}^{(4)} = -J\sum_{\sigma,j}\left[c^{\dagger}_{\sigma,j}c_{\sigma,j+1}\left(-n_{-\sigma,j} - n_{-\sigma,j+1}\right) + c^{\dagger}_{\sigma,j}c^{\dagger}_{\sigma,j+1}\left(n_{-\sigma,j} - n_{-\sigma,j+1}\right)\right] + h.c.$$

spectrum obtained from finite DMRG L=16



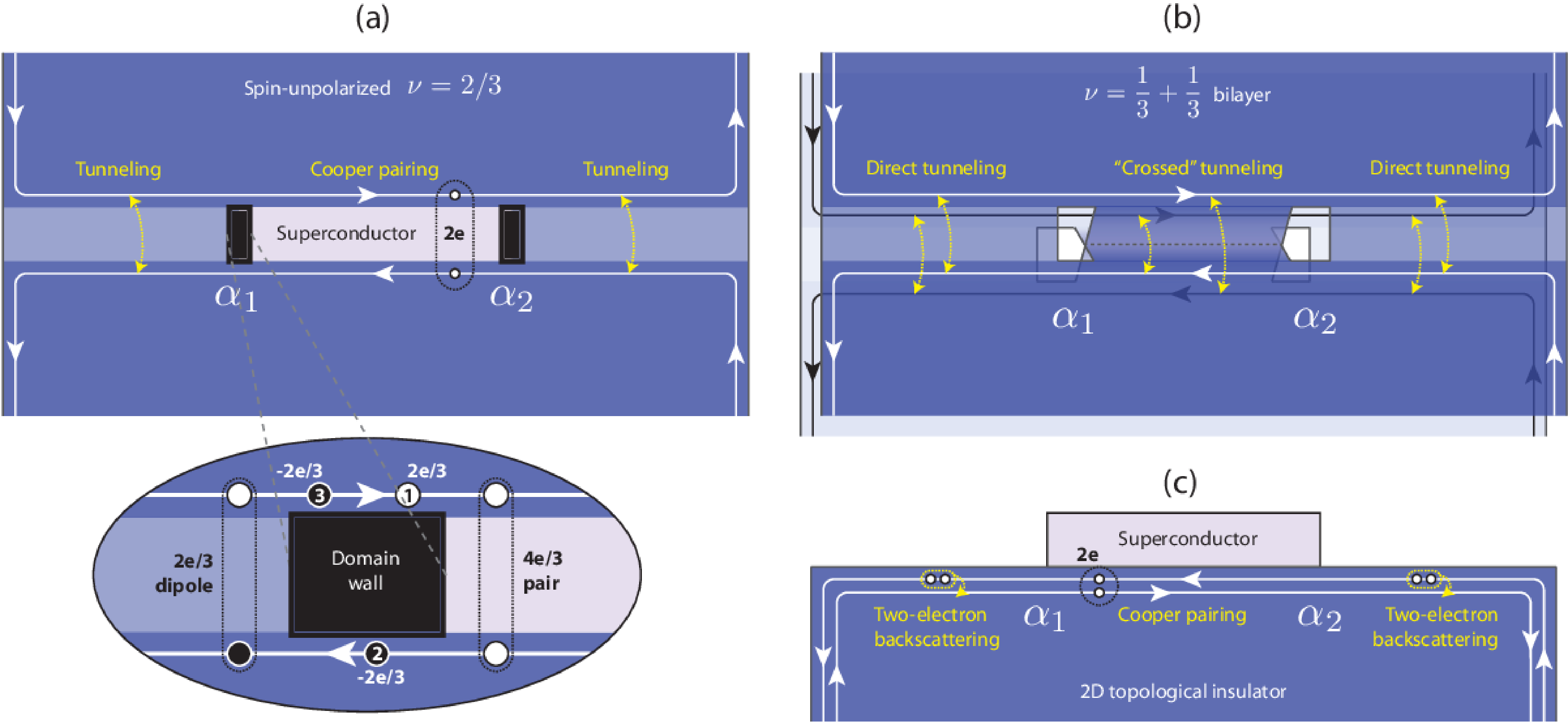
Parafermions in nanowires:

superconductivity

+

occupation dependent hopping

Possible Experimental realizations



J. Alicea, P. Fendley Annu. Rev. Condens. Matter Phys. 7,119–39 (2016.)

Majoranas vs. parafermions

- Majorana modes can be potentially realized in non-interacting systems. (*i.e.* mean-field description is sufficient)
- With braiding alone, Majorana modes can realize nontrivial unitary operations, but no entangling qbit gates.
- Parafermions need interaction. (*i.e.* mean-field description is not sufficient)
- $\mathbb{Z}_{even}$ parafermions can realize entangling gates just with braiding!

A.Hutter, D. Loss Phys. Rev. B **93**, 125105 (2016)

Further reading

- J. Alicea, P. Fendley: Topological Phases with Parafermions: Theory and Blueprints; Annu. Rev. Condens. Matter Phys. **7**,119 (2016)

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