Some practical notes on the Jordan-Wigner transformation

Spin-$\frac{1}{2}$to Majorana

Consider the spin operator algebra of spin-$\frac{1}{2}$ entities.

$$\left(\hat{\sigma}_j^z\right)^2 = \left(\hat{\sigma}_j^x\right)^2 = \hat{1}, \quad \hat{\sigma}_j^x\hat{\sigma}_j^z = -\hat{\sigma}_j^z\hat{\sigma}_j^x, \quad \hat{\sigma}_q^x\hat{\sigma}_p^z = \hat{\sigma}_p^z\hat{\sigma}_q^x \; (p \neq q) \tag{1}$$

The Jordan-Wigner (Kadanoff-Fradkin) transformation builds from these

$$\gamma_{2p-1} = \hat{\sigma}_p^z \prod_{q<p} \hat{\sigma}_q^x = \hat{\sigma}_p^z \hat{W}_p \tag{2}$$

$$\gamma_{2p} = \mathrm{i}\hat{\sigma}_p^x\hat{\sigma}_p^z\hat{W}_p \tag{3}$$

$$\left[\hat{\sigma}_p^{x/z}, \hat{W}_p\right] = 0 \tag{4}$$

$$\gamma_{2p-1}\gamma_{2p-1} = \hat{\sigma}_p^z\hat{W}_p\hat{\sigma}_p^z\hat{W}_p = \hat{1} \tag{5}$$

since every constituent commute.

$$\gamma_{2p}\gamma_{2p} = \left(\mathrm{i}\hat{\sigma}_p^x\hat{\sigma}_p^z\hat{W}_p\right)\left(\mathrm{i}\hat{\sigma}_p^x\hat{\sigma}_p^z\hat{W}_p\right) = \hat{1} \tag{6}$$

since $\mathrm{i}^2 = -1$ and $\hat{\sigma}_j^x\hat{\sigma}_j^z = -\hat{\sigma}_j^z\hat{\sigma}_j^x$. Similarly follows that $\gamma_p^\dagger = \gamma_p$, since all constituents are Hermitian and they either commute or if not complex conjugation of the extra factor of i balances the negative sign from the anticommutator.

In order to see how two nonequivalent γ-s behave we can assume, without the loss of generality, $k < l$. Then

$$\gamma_{2k-1}\gamma_{2l-1} = \hat{\sigma}_k^z\hat{W}_k\hat{\sigma}_l^z\hat{W}_l = \hat{\sigma}_k^z\hat{W}_k\hat{\sigma}_l^z\hat{W}\hat{W}_k = -\hat{\sigma}_l^z\hat{W}\hat{W}_k\hat{\sigma}_k^z\hat{W}_k = -\gamma_{2l-1}\gamma_{2k-1} \tag{7}$$

where $W = \prod_{q=k}^{l-1} \hat{\sigma}_q^x$ and since it contains $\hat{\sigma}_k^x$ we gain an extra factor of -1 when we commute it with $\hat{\sigma}_k^z$. Note that the same reasoning could have been be applied if we would have chosen different combination of Majoranas since in all cases one has to swap W with $\hat{\sigma}_k^z$. In order to rewrite the Hamiltonian

$$H = -J\sum_{p=1}^{L-1} \hat{\sigma}_p^z\hat{\sigma}_{p+1}^z - f\sum_{p=1}^{L-1} \hat{\sigma}_p^x \tag{8}$$

we first note that

$$\gamma_{2p}\gamma_{2p-1} = \mathrm{i}\hat{\sigma}_p^x\hat{\sigma}_p^z\hat{W}_p\hat{\sigma}_p^z\hat{W}_p = \mathrm{i}\hat{\sigma}_p^x \tag{9}$$

since everything commutes beyond $\hat{\sigma}_p^x$. Thus we can rewrite the second term of the Hamiltonian with $\hat{\sigma}_p^x = \mathrm{i}\gamma_{2p-1}\gamma_{2p}$. Focusing now on the first term

$$\gamma_{2p}\gamma_{2(p+1)-1} = \mathrm{i}\hat{\sigma}_p^x\hat{\sigma}_p^z\hat{W}_p\hat{\sigma}_{p+1}^z\hat{W}_{p+1} = \mathrm{i}\hat{\sigma}_p^x\hat{\sigma}_p^z\hat{W}_p\hat{\sigma}_{p+1}^z\hat{W}_p\hat{\sigma}_p^x, \tag{10}$$

here almost all terms commute except of course $\hat{\sigma}_p^x$ and $\hat{\sigma}_p^z$, thus interchanging them at a cost of an extra -1 factor we get

$$\gamma_{2p}\gamma_{2(p+1)-1} = -\mathrm{i}\hat{\sigma}_p^z\hat{\sigma}_{p+1}^z, \tag{11}$$

thus

$$H = -J\mathrm{i}\sum_{p=1}^{L-1} \gamma_{2p}\gamma_{2(p+1)-1} - f\mathrm{i}\sum_{p=1}^{L-1} \gamma_{2p-1}\gamma_{2p}. \tag{12}$$

Clocks to Parafermions

Consider the 1D clock Hamiltonian

$$H = -J\mathrm{e}^{\mathrm{i}\phi}\sum_{p=1}^{L-1} \hat{\sigma}_p^\dagger\hat{\sigma}_{p+1} - f\mathrm{e}^{\mathrm{i}\theta}\sum_{p=1}^{L-1} \hat{\tau}_p + \text{h.c.} \tag{13}$$

with operators $\hat{\sigma}_p$ and $\hat{\tau}_p$ satisfying the algebra

$$\hat{\sigma}_p^N = \hat{\tau}_p^N = \hat{1}, \tag{14}$$

$$\hat{\sigma}_p^{N-1} = \hat{\sigma}_p^\dagger, \hat{\tau}_p^{N-1} = \hat{\tau}_p^\dagger \tag{15}$$

$$\hat{\sigma}_p\hat{\tau}_p = \omega\hat{\tau}_p\hat{\sigma}_p, \tag{16}$$

$$\hat{\sigma}_p\hat{\tau}_q = \hat{\tau}_q\hat{\sigma}_p, \; p \neq q \tag{17}$$

$$\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}}, \quad \varpi = -\mathrm{e}^{\mathrm{i}\frac{\pi}{N}}. \tag{18}$$

Building on these operators we can define, in the same spirit as before the Jordan-Wigner strings as

$$\hat{\alpha}_{2p-1} = \hat{\sigma}_p \prod_{q<p} \hat{\tau}_q = \hat{\sigma}_p \hat{K}_p, \tag{19}$$

$$\hat{\alpha}_{2p} = \varpi \hat{\tau}_p \hat{\sigma}_p \hat{K}_p. \tag{20}$$

As before we note again

$$\left[\hat{\sigma}_p, \hat{K}_p\right] = \left[\hat{\tau}_p, \hat{K}_p\right] = 0, \tag{21}$$

Regarding the algebra of the operators $\hat{\alpha}_p$ we prove some identities

$$\hat{\alpha}_{2p-1}^N = \left[\hat{\sigma}_p \hat{K}_p\right]^N = 1 \tag{22}$$

since all constituents commute and thus we can take their individual powers, the other parafermions require a bit more steps but ultimate yield the same:

$$\hat{\alpha}_{2p}^N = \left[\varpi \hat{\tau}_p \hat{\sigma}_p \hat{K}_p\right]^N = \varpi^N \left[\hat{\tau}_p \hat{\sigma}_p\right]^N \hat{K}_p^N = \varpi^N \left[\hat{\tau}_p \hat{\sigma}_p\right]^N = \varpi^N \omega^{\frac{N(N+1)}{2}} = \left(\varpi \omega^{\frac{N(N+1)}{2}}\right)^N = \left(-\mathrm{e}^{\mathrm{i}\frac{\pi}{N}} \mathrm{e}^{\mathrm{i}\frac{\pi}{N}(N+1)}\right)^N = \left(-\mathrm{e}^{\mathrm{i}\pi} \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}}\right)^N = 1. \tag{23}$$

Relation to the adjoint operators can be formulated as

$$\hat{\alpha}_{2p-1}^\dagger = \hat{\sigma}_p^\dagger \hat{K}_p^\dagger = \hat{\alpha}_{2p-1}^{N-1}, \tag{24}$$

again using the commutation of clock operators on different sites, while a bit more steps again give $\hat{\alpha}_{2p}^\dagger = \hat{\alpha}_{2p}^{N-1}$. For the commutation relations we again assume $k < l$ and look at

$$\hat{\alpha}_{2k-1}\hat{\alpha}_{2l-1} = \hat{\sigma}_k \hat{K}_k \hat{\sigma}_l \hat{K}_l = \hat{\sigma}_k \hat{K}_k \hat{\sigma}_l \hat{K}_k \hat{K} = \omega \hat{\sigma}_l \hat{K}_k \hat{K} \hat{\sigma}_k \hat{K}_k \tag{25}$$

where in the last step exchanging $\hat{K} = \prod_{q=k}^{l-1} \hat{\tau}_q$ and $\hat{\sigma}_k$ comes now at a cost of a factor ω! A similar reasoning for all combinations yields the same. Thus we have

$$\alpha_k \alpha_l = \omega^{\mathrm{sgn}(l-k)} \alpha_l \alpha_k. \tag{26}$$

This is the anyonic comutation relation obeyed by parafermions!

We now rewrite the clock Hamiltonian with parafermions. Focusing again first on the second term we note that

$$\alpha_{2p-1}^\dagger \alpha_{2p} = \left(\hat{\sigma}_p \hat{K}_p\right)^\dagger \varpi \hat{\tau}_p \hat{\sigma}_p \hat{K}_p = \varpi \hat{\sigma}_p^\dagger \hat{\tau}_p \hat{\sigma}_p = \varpi \omega \hat{\tau}_p, \tag{27}$$

thus

$$\hat{\tau}_p = \varpi \alpha_{2p-1}^\dagger \alpha_{2p}. \tag{28}$$

In order to obtain the first term we look at

$$\begin{aligned} \alpha_{2p}^\dagger \alpha_{2p+1} &= \left(\varpi \hat{\tau}_p \hat{\sigma}_p \hat{K}_p\right)^\dagger \hat{\sigma}_{p+1} \hat{K}_{p+1} \\ &= \varpi^* \hat{\sigma}_p^\dagger \hat{\tau}_p^\dagger \hat{\sigma}_{p+1} \hat{\tau}_p = \varpi^* \hat{\sigma}_p^\dagger \hat{\sigma}_{p+1} \end{aligned} \tag{29}$$

$$\hat{\sigma}_p^\dagger \hat{\sigma}_{p+1} = \varpi \alpha_{2p}^\dagger \alpha_{2p+1} \tag{30}$$

thus the Hamiltonian can be written as

$$H = -J\mathrm{e}^{\mathrm{i}\phi} \sum_{p=1}^{L-1} \varpi \alpha_{2p}^\dagger \alpha_{2p+1} - f\mathrm{e}^{\mathrm{i}\theta} \sum_{p=1}^{L-1} \varpi \alpha_{2p-1}^\dagger \alpha_{2p} + \text{h.c.} \tag{31}$$