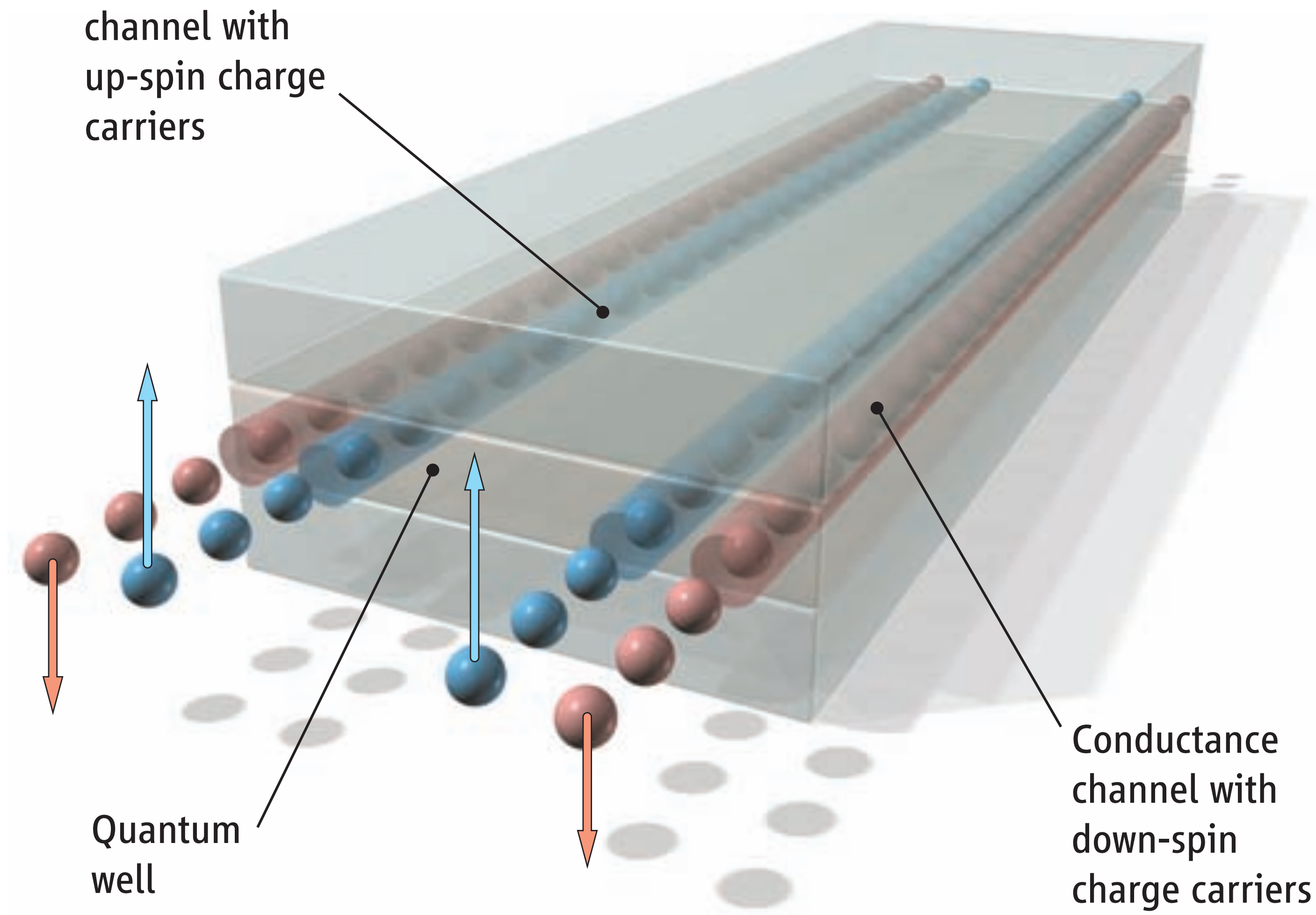
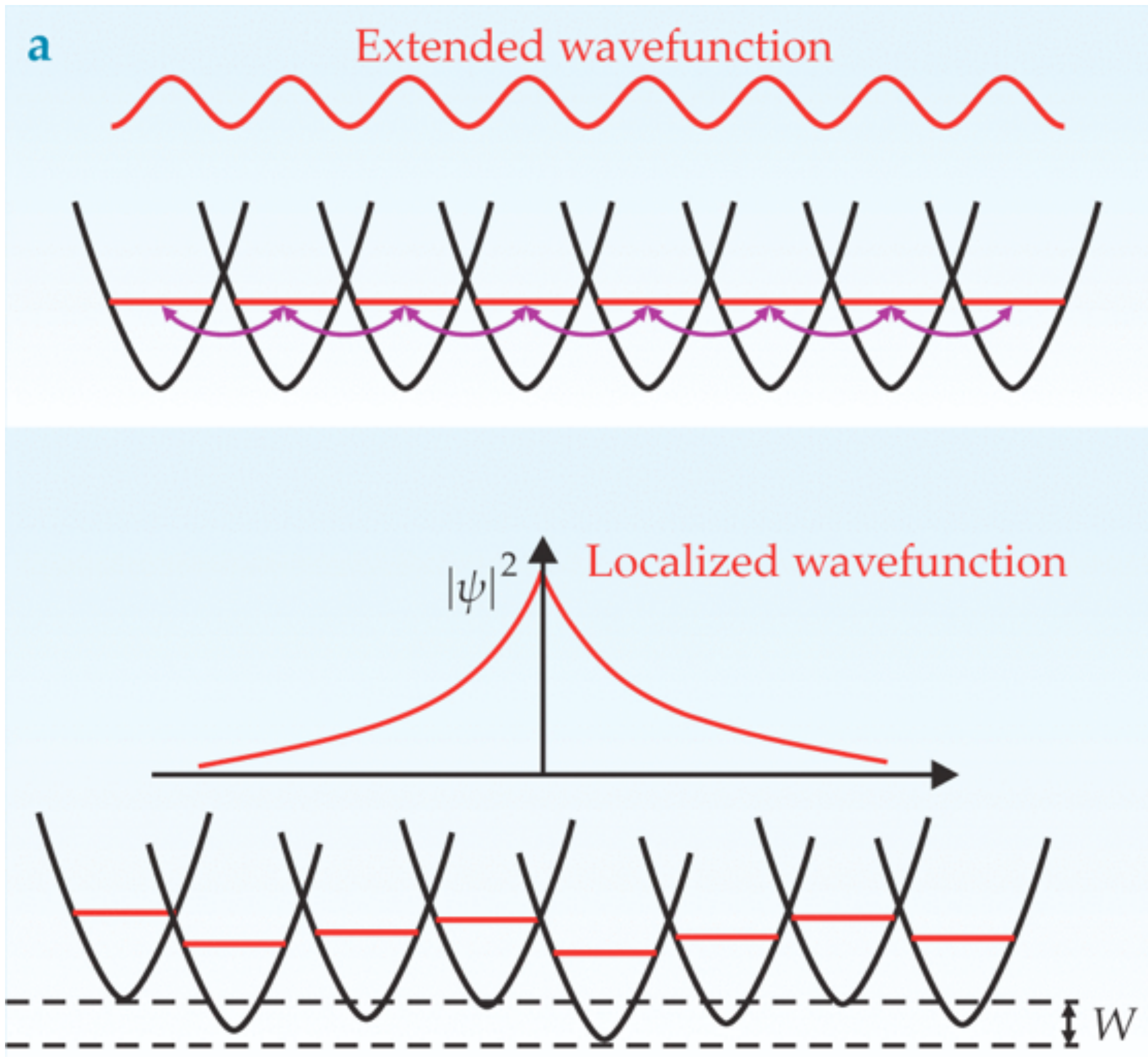
Chapter 8

Time-reversal symmetric two-dimensional topological insulators – the Bernevig–Hughes–Zhang model



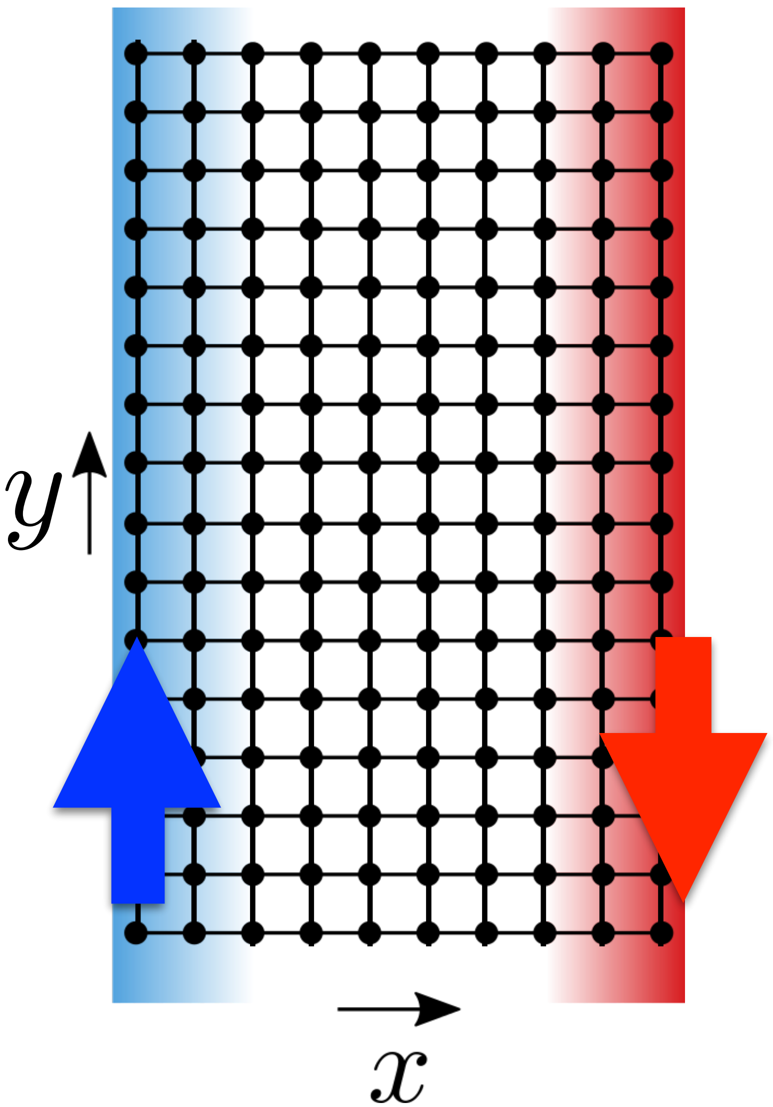
Schematic of the spin-polarized edge channels in a quantum spin Hall insulator.

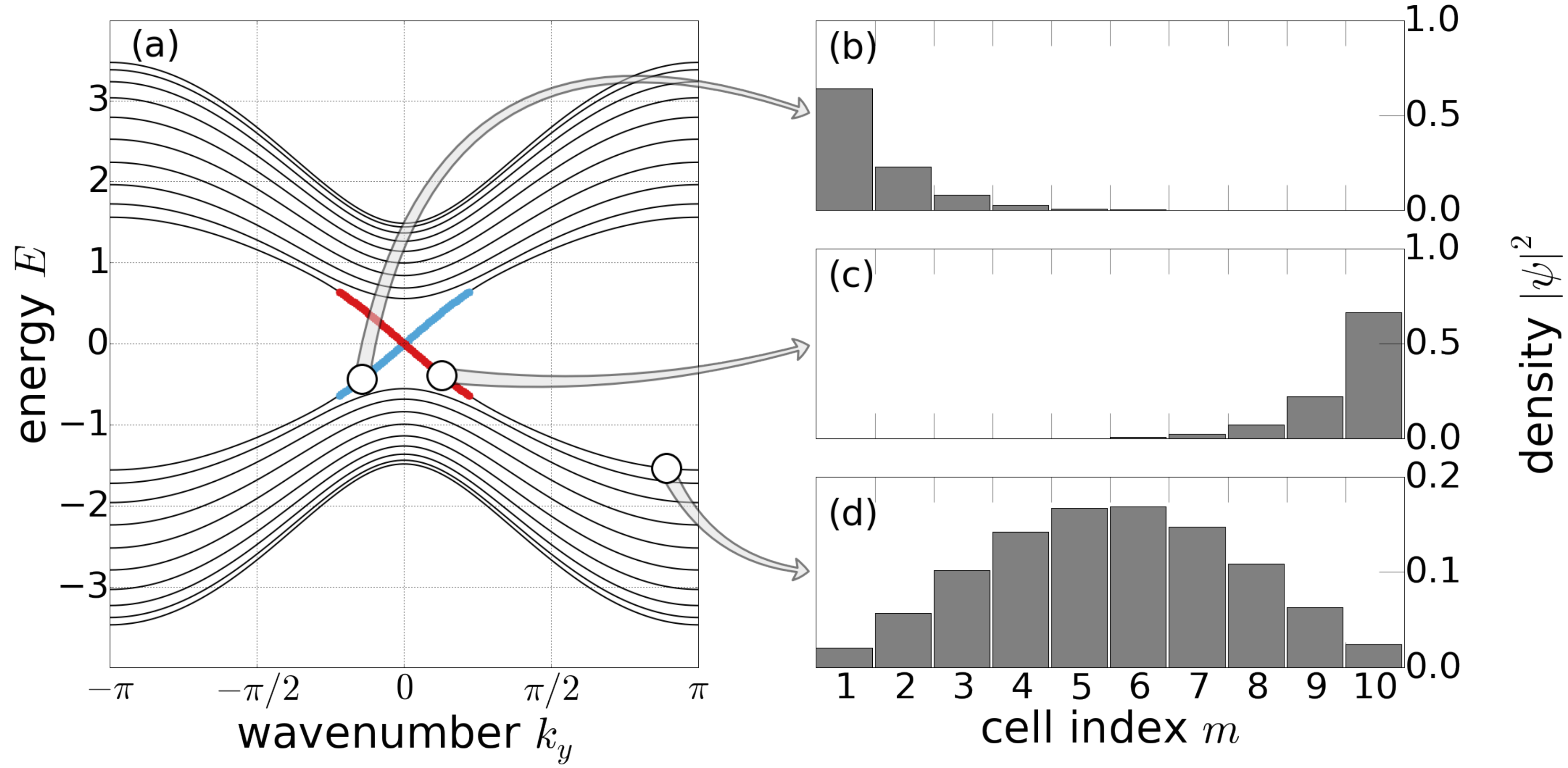
Anderson localization in 1D



In 1D, even a tiny disorder renders the wavefunctions localized.
Hence, disorder transforms a metal into an (Anderson) insulator.

1D edge of 2D Chern insulator: no localization





Fermi level in gap
=> edge electrons can't be backscattered
=> **edge conductor** (quantized conductance)

Qi et al.
PRB 2006

1D edge of 2D time-reversal invariant topological insulator: edge states conduct or localize?

They conduct.

(Fermionic) Time-Reversal Symmetry

usual symmetry:

U unitary such that $UHU^{-1} = H$.

chiral symmetry:

Γ unitary such that $\Gamma H \Gamma^{-1} = -H$

(fermionic) time-reversal symmetry:

$\mathcal{T}$ antiunitary such that $\mathcal{T}^2 = -1$, and $\mathcal{T} H \mathcal{T}^{-1} = H$.

(bosonic) time-reversal symmetry:

$\mathcal{T}$ antiunitary such that $\mathcal{T}^2 = +1$, and $\mathcal{T} H \mathcal{T}^{-1} = H$.

… from now on, TRS means fermionic TRS.

Kramers degeneracy in the band structure

Kramers theorem:

Take a Hamiltonian with fermionic time-reversal symmetry $\mathcal{T}$.

Take an eigenstate $|\psi\rangle$ of H with energy E.

Then, $\mathcal{T}\,|\psi\rangle$ is also an energy eigenstate with energy E, and $\langle\psi|\mathcal{T}\psi\rangle = 0$.

Consequence for band structures:

In a crystal with fermionic time-reversal symmetry, every band is twofold degenerate at time-reversal-invariant momenta.

BHZ model: edge-state Kramers pairs, robust against time-reversal-invariant perturbation

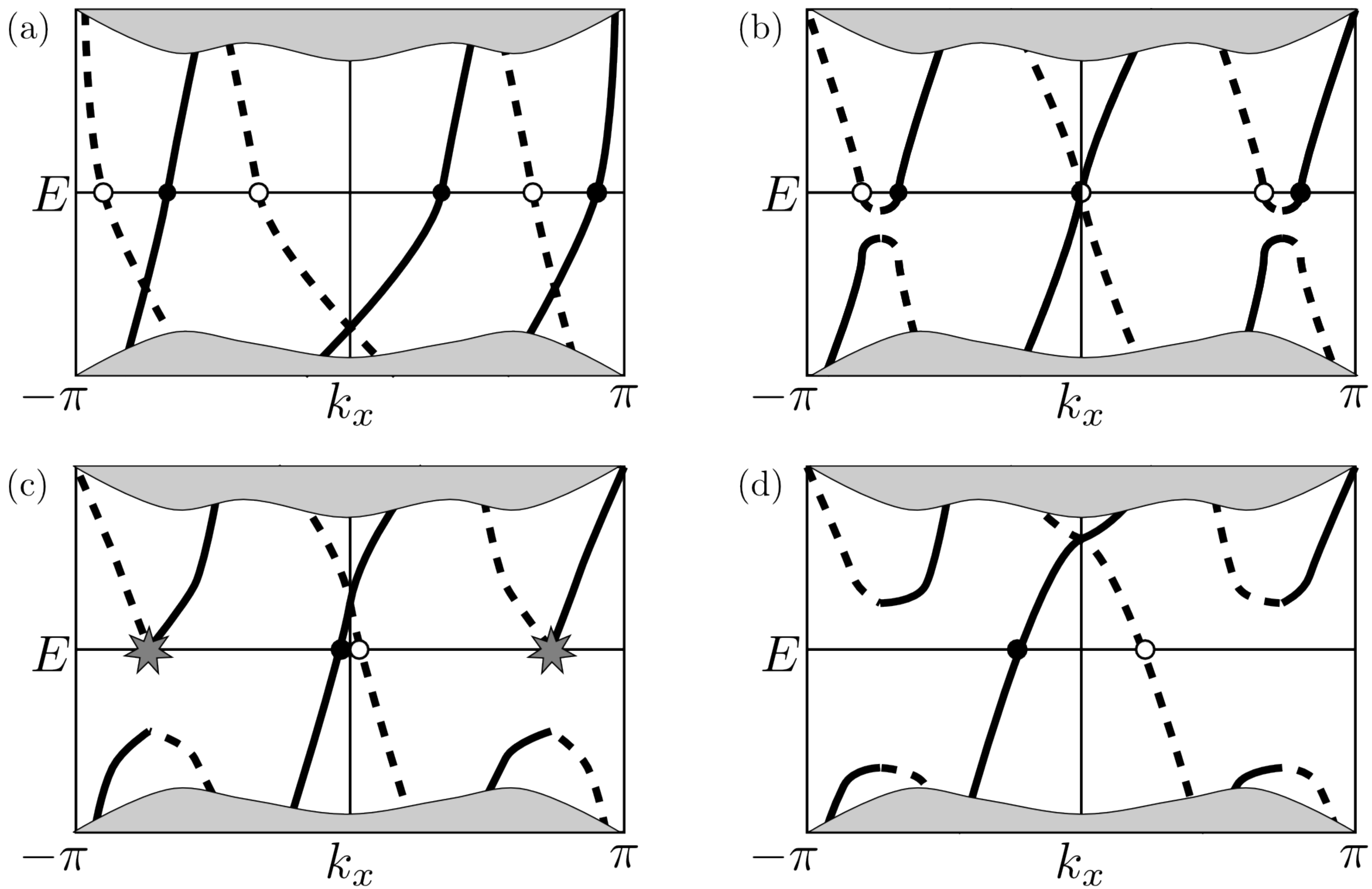
example: Bernevig-Hughes-Zhang model

$$\hat{H}_{\mathrm{BHZ}}(\mathbf{k}) = \hat{s}_0 \otimes [(u + \cos k_x + \cos k_y)\hat{\boldsymbol{\sigma}}_z + \sin k_y \hat{\boldsymbol{\sigma}}_y)] + \hat{s}_z \otimes \sin k_x \hat{\boldsymbol{\sigma}}_x + \hat{s}_x \otimes \hat{C},$$

$$\hat{H}_{\mathrm{BHZ}} = \hat{s}_0 \otimes [(u + \cos k_x + \cos k_y)\hat{\sigma}_z + \sin k_x \hat{\sigma}_x] + \hat{s}_z \otimes \sin k_y \hat{\sigma}_y + \hat{s}_x \otimes \hat{C}.$$

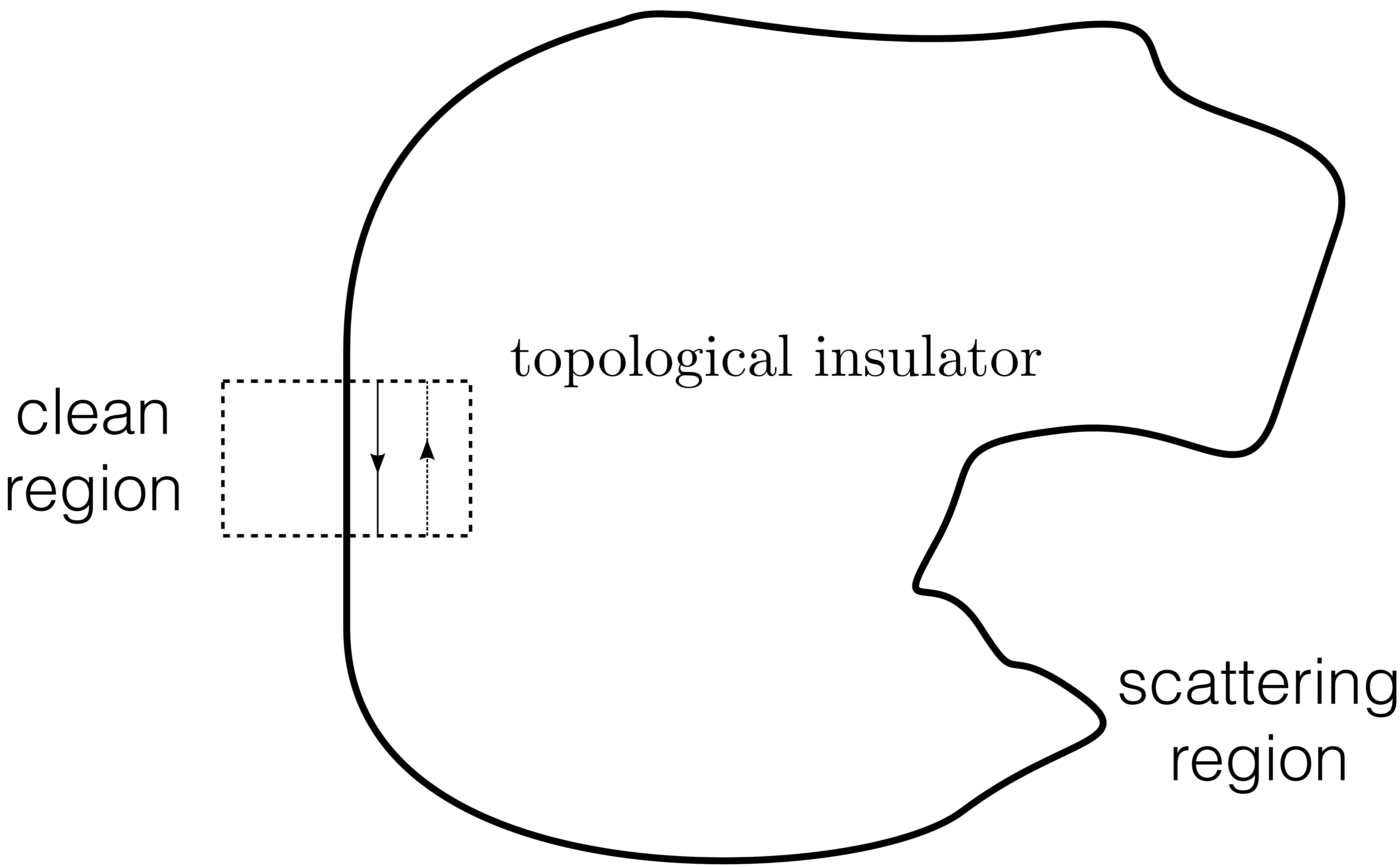
Fig. 8.1 Stripe dispersion relations of the BHZ model, with sublattice potential parameter $u = -1.2$. Right/left edge states (more than 60% weight on the last/first two columns of unit cells) marked in dark red/light blue. (a): uncoupled layers, $\hat{C} = 0$. (c): Antisymmetric coupling $\hat{C} = 0.3\sigma_y$ cannot open a gap in the edge spetrum.

Number parity of edge-state Kramers pairs is a topological invariant



… from now on, *topological insulator* refers to
2D topological insulator with fermionic time-reversal symmetry

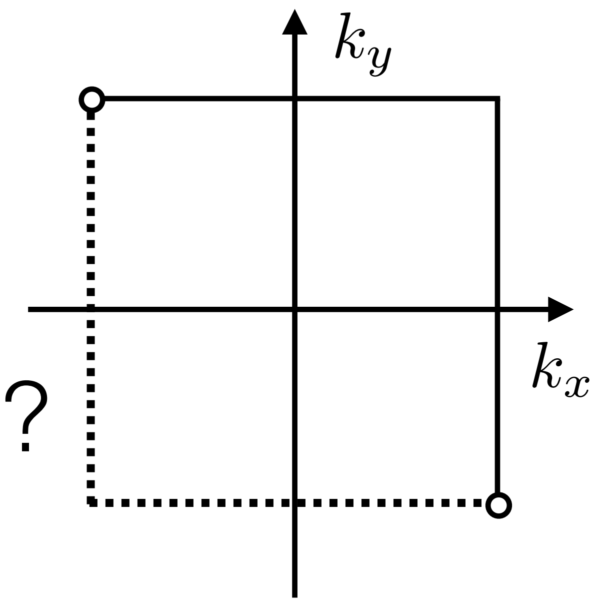
Absence of backscattering



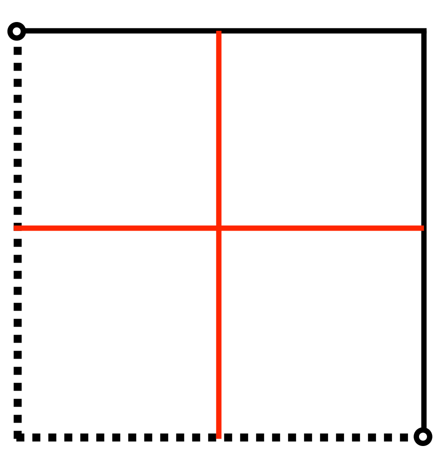
Full transmission through scattering region => absence of localization

Time-reversal invariant momenta (1)

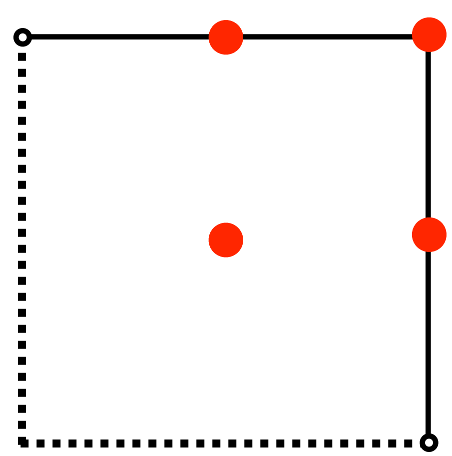
Take a 2D square crystal lattice.
Its Brillouin zone is sketched here.
Where are the time-reversal invariant momenta?



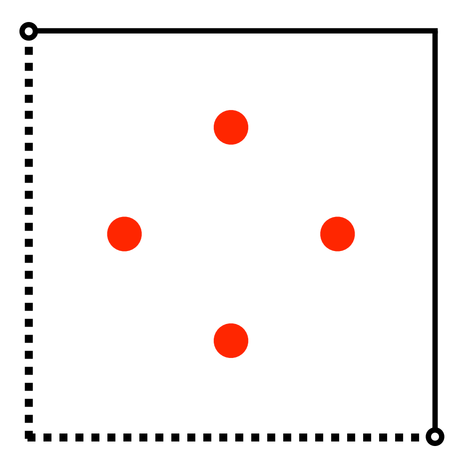
A



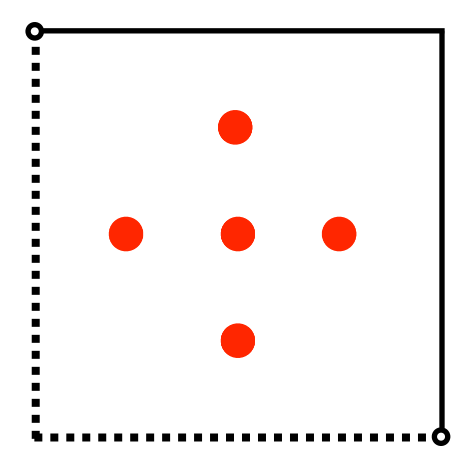
B



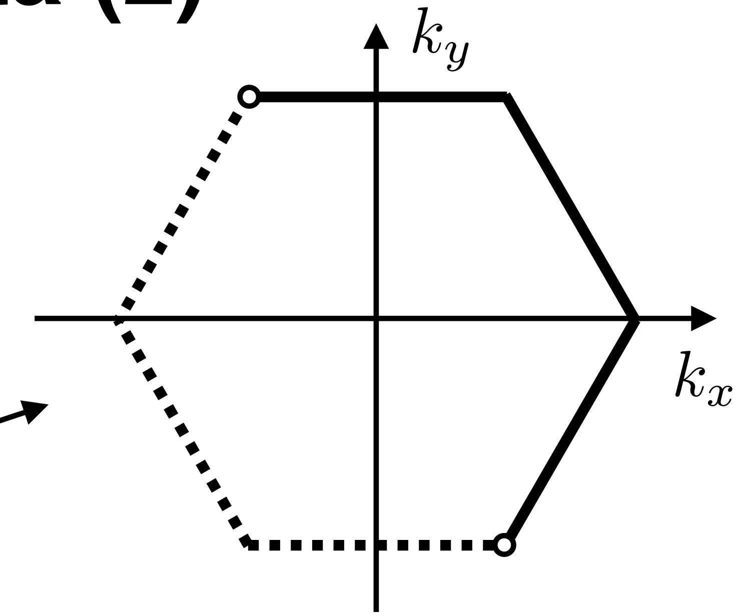
C



D

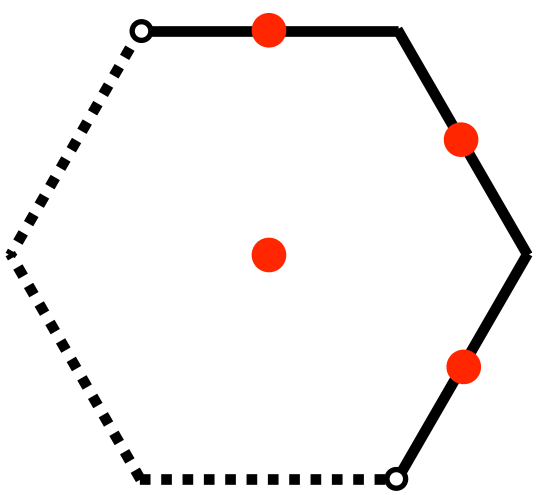


Time-reversal invariant momenta (2)



Take a 2D triangular crystal lattice.
Its Brillouin zone is sketched here.
Where are the time-reversal invariant momenta?

A



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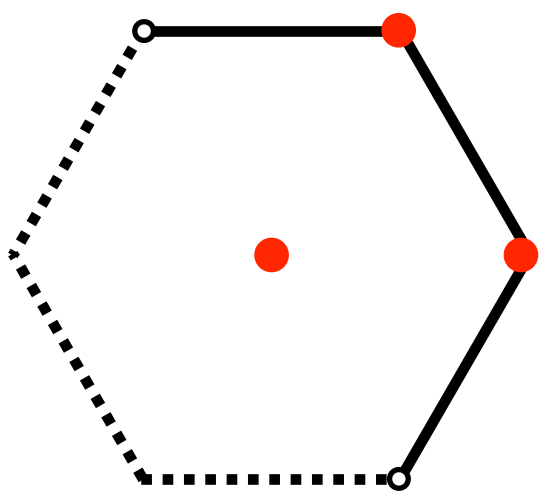

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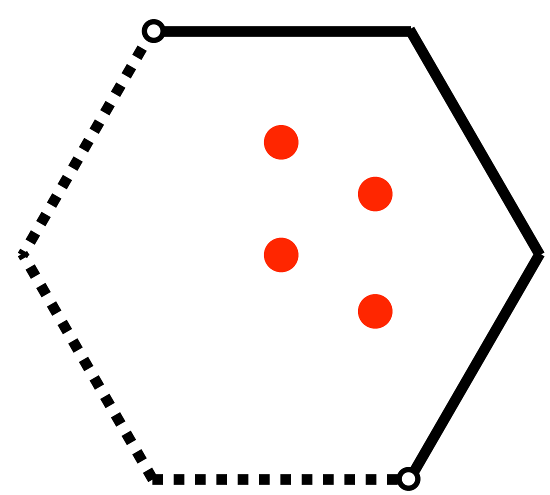
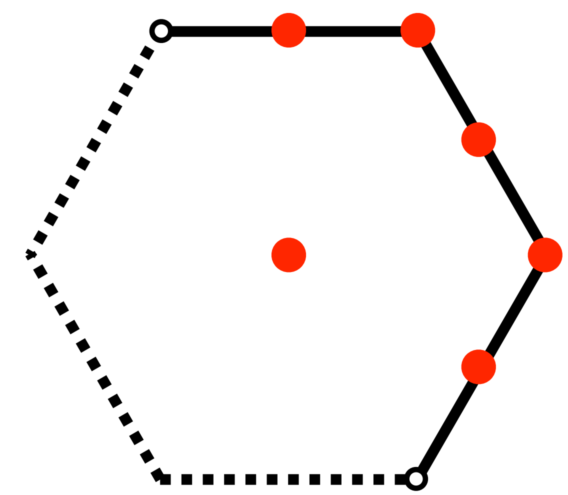
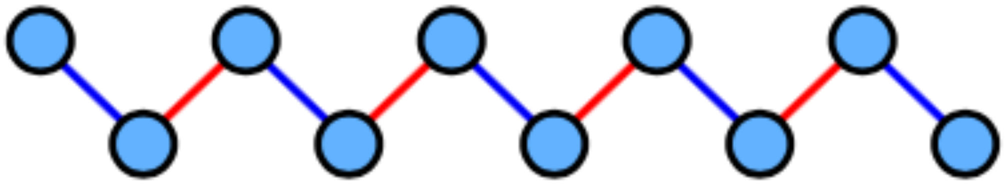


D



SSH model with nearest-neighbor real-valued hopping



Consider the SSH model with only nearest-neighbor hopping, all hopping amplitudes being real.
Does it have time-reversal symmetry?

(a) Yes, it has fermionic time-reversal symmetry.

(b) Yes, it has bosonic time-reversal symmetry.

(c) Yes, both types.

(d) No.

Edge states and time reversal symmetry (1)

Take a clean QWZ model with Chern number 1.
Consider the edge state $|\Psi\rangle$ with a given wave number k.

Then $\mathcal{T}\,|\Psi\rangle$ ...

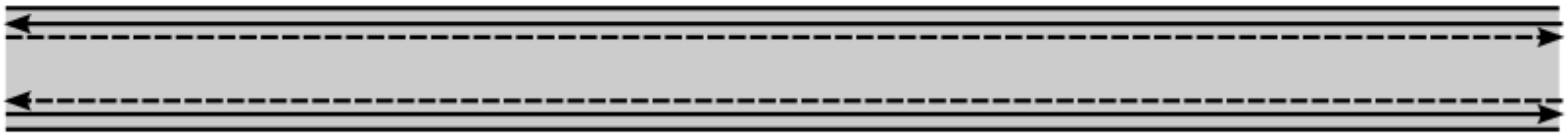
(a) ... is orthogonal to $|\Psi\rangle$, and is an eigenstate of the Hamiltonian with the same energy as $|\Psi\rangle$.

(b) ... is orthogonal to $|\Psi\rangle$, and is an eigenstate of the Hamiltonian which propagates on the other edge.

(c) ... is orthogonal to $|\Psi\rangle$, but it is not an eigenstate of the Hamiltonian.

(d) ... doesn't exist: time reversal can't be applied as the model has no time-reversal symmetry.

Edge states and time reversal symmetry (2)



Take an edge state $|\Psi\rangle$ of a clean topological insulator with a given wave number k.

Then $\mathcal{T}\,|\Psi\rangle$ ...

(a) ... is orthogonal to $|\Psi\rangle$, and is an eigenstate of the Hamiltonian with the same energy as $|\Psi\rangle$.

(b) ... is orthogonal to $|\Psi\rangle$, and is an eigenstate of the Hamiltonian which propagates on the other edge.

(c) ... is orthogonal to $|\Psi\rangle$, but it is not an eigenstate of the Hamiltonian.

(d) ... is the same as $|\Psi\rangle$, since the system has time-reversal symmetry.

Two-band model with time reversal symmetry

2D two-band lattice models with fermionic time-reversal symmetry ...

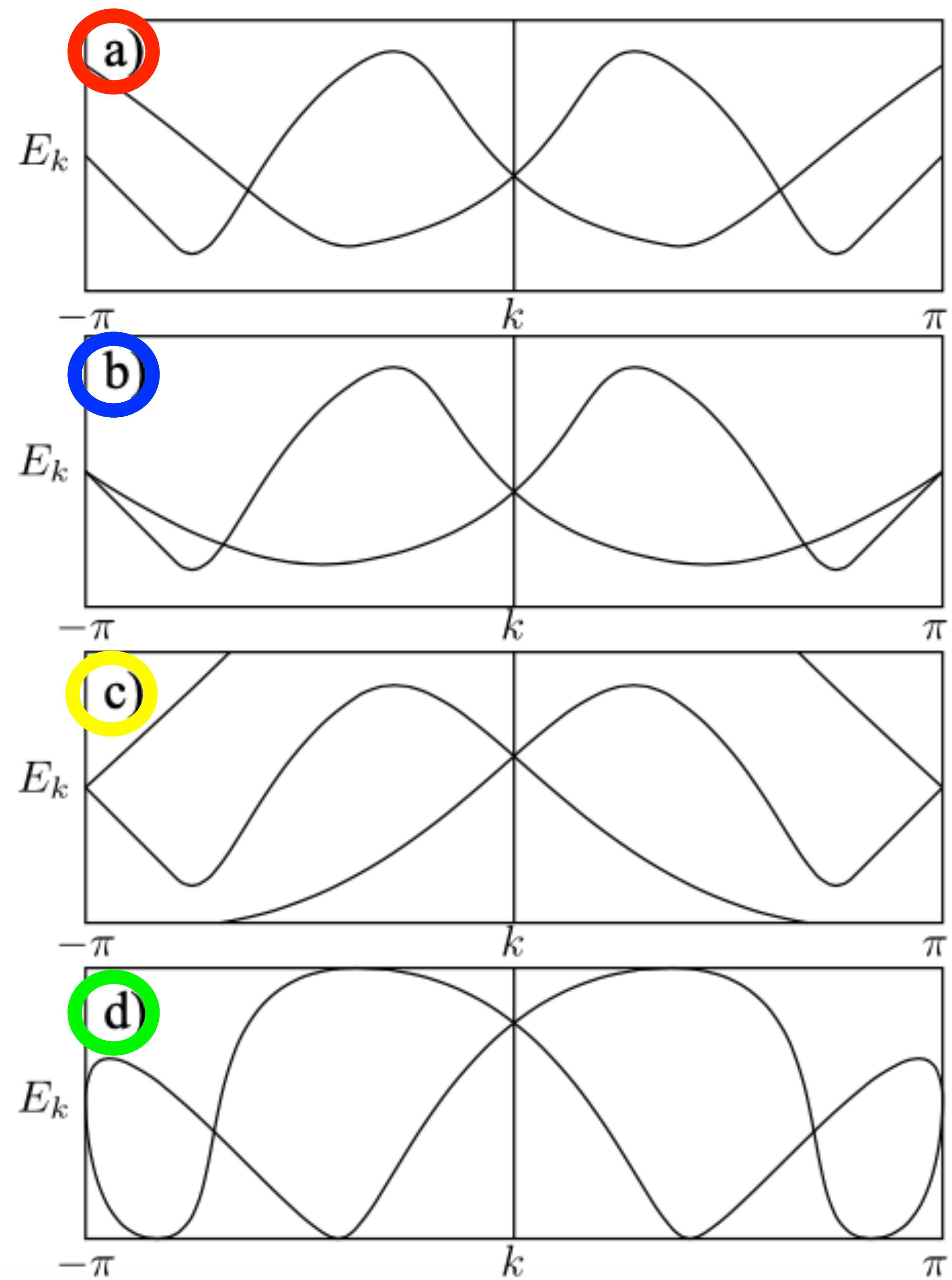
a) ... always have a band gap.

b) ... never have a band gap.

c) ... might have a band gap.

d) ... do not exist.

Edge spectrum of a 2D topological insulator



Each figure shows the edge spectrum of a 2D insulator.

Bulk valence bands are way below, and bulk conduction bands are way above these edge states.

Which edge spectrum belongs to a 2D **topological** insulator?

Scattering in a topological insulator (1)

Consider a long and wide ribbon of a 2D topological insulator, in which each edge hosts a single edge-state Kramers pair. Part of the ribbon is disordered and serves as a scattering region. The whole system is time-reversal symmetric.

What is the dimension of the scattering matrix S describing the scattering region?

(a) 2×2

(b) 4×4

(c) 8×8

(d) 16×16

Scattering in a topological insulator (2)

Consider a long and wide ribbon of a 2D topological insulator, in which each edge hosts a single edge-state Kramers pair. Part of the ribbon is disordered and serves as a scattering region. The whole system is time-reversal symmetric.

How many nonzero entries are there in the scattering matrix S of the scattering region?

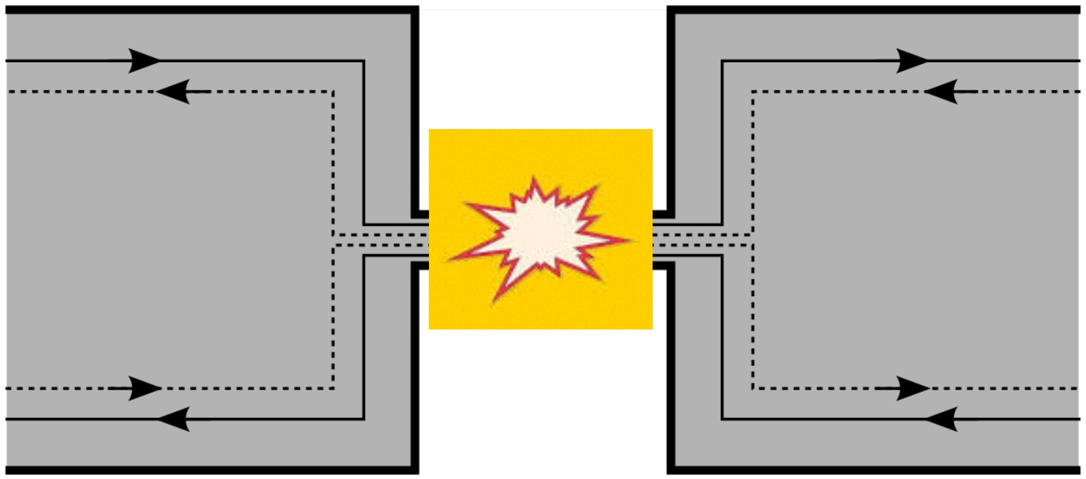
(a) 0

(b) 2

(c) 4

(d) 8

Scattering in a topological insulator (3)



Consider a constriction of a ribbon of a 2D topological insulator, in which each edge hosts a single edge-state Kramers pair. The constriction is disordered and serves as a scattering region. The whole system is time-reversal symmetric.

How many zero entries are guaranteed in the scattering matrix S of the constriction?

(a) 0

(b) 4

(c) 8

(d) 12