HUNGARIAN ACADEMY OF SCIENCES
WIGNER RESEARCH CENTRE FOR PHYSICS

Introduction to topological insulators

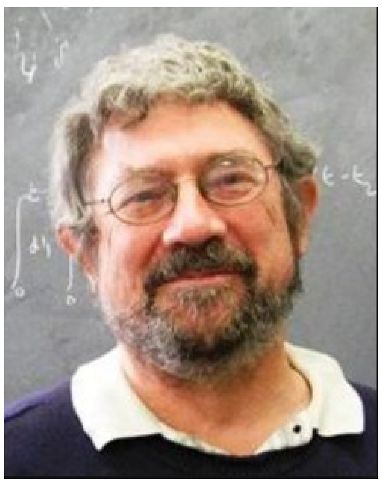
Janos Asboth[1,3], Laszlo Oroszlany[2], Andras Palyi[3]

1: Wigner Research Centre for Physics, Hungarian Academy of Sciences
2: Eotvos University, Budapest
3: Technical University, Budapest

Budapest, 2019 September 10

2016 Nobel prize in physics: 3 British scientists, "Theoretical discovery of topological phases and phase transitions"

1/4



J M Kosterlitz
*1942, Scotland
PhD: Oxford

1/2

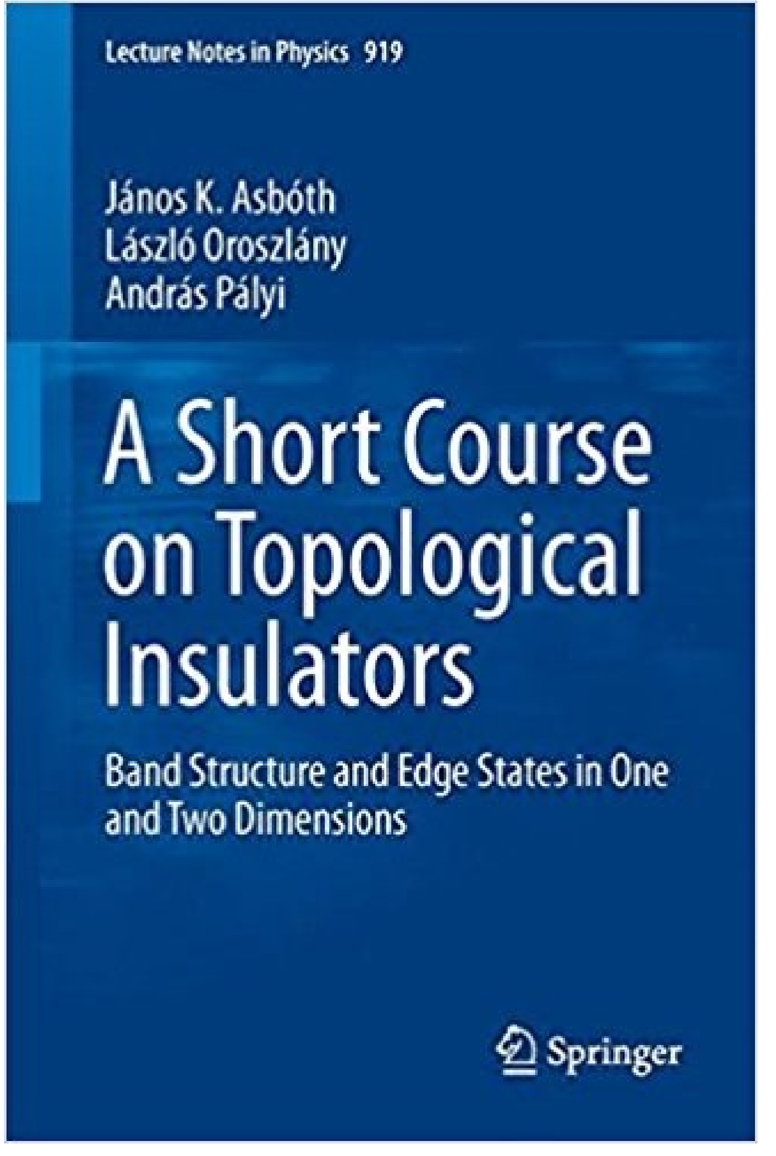
D J Thouless
*1934, Scotland
PhD: Cornell,
Advisor: Bethe

1/4

F D Haldane
*1951, London
PhD: Cambridge,
Advisor: Anderson

Technical details of the course

- 1+12 lectures
- Book: A Short Course on Topological Insulators: Band-structure topology and edge states in one and two dimensions
- On arxiv
- End of semester: written + oral exam for grade

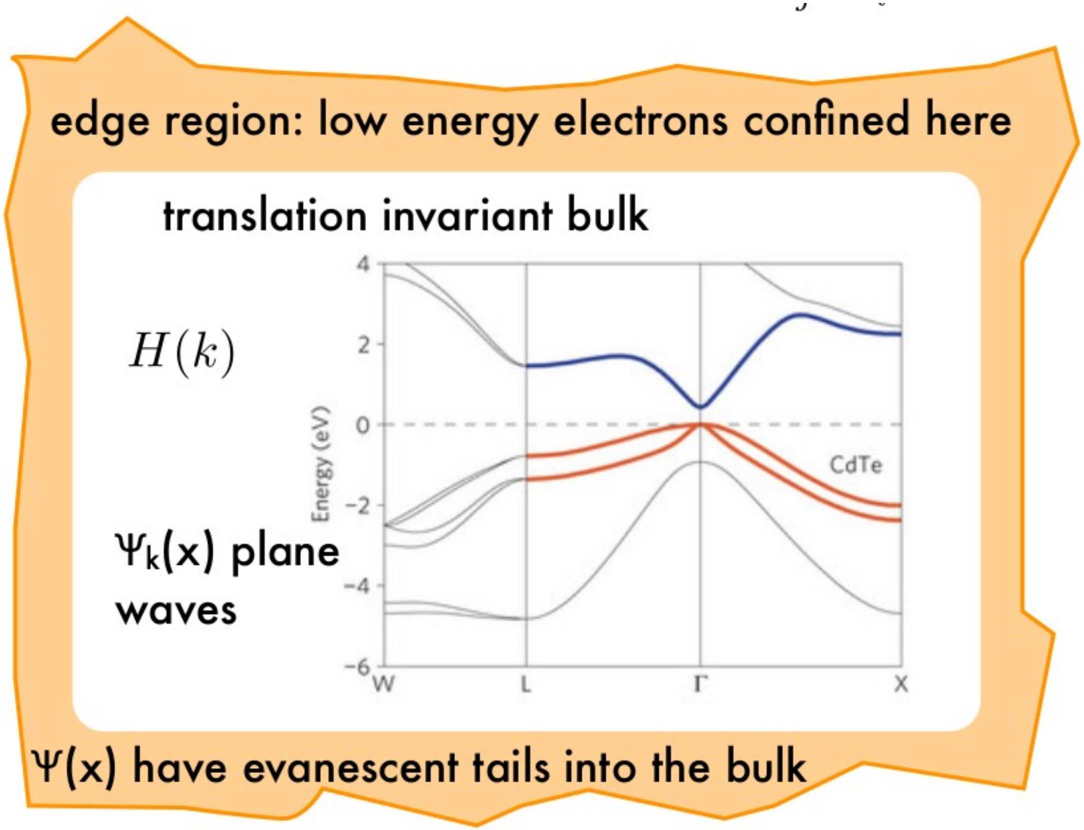


- The book and extra material downloadable from eik.bme.hu/~palyi/TopologicalInsulators2017Fall/

Insulator: has bulk energy gap separating fully occupied bands from fully empty ones

$$\hat{H} = \sum_{\langle xx' \rangle} H_{xx'} \hat{c}^{\dagger}_{x'} \hat{c}_x$$

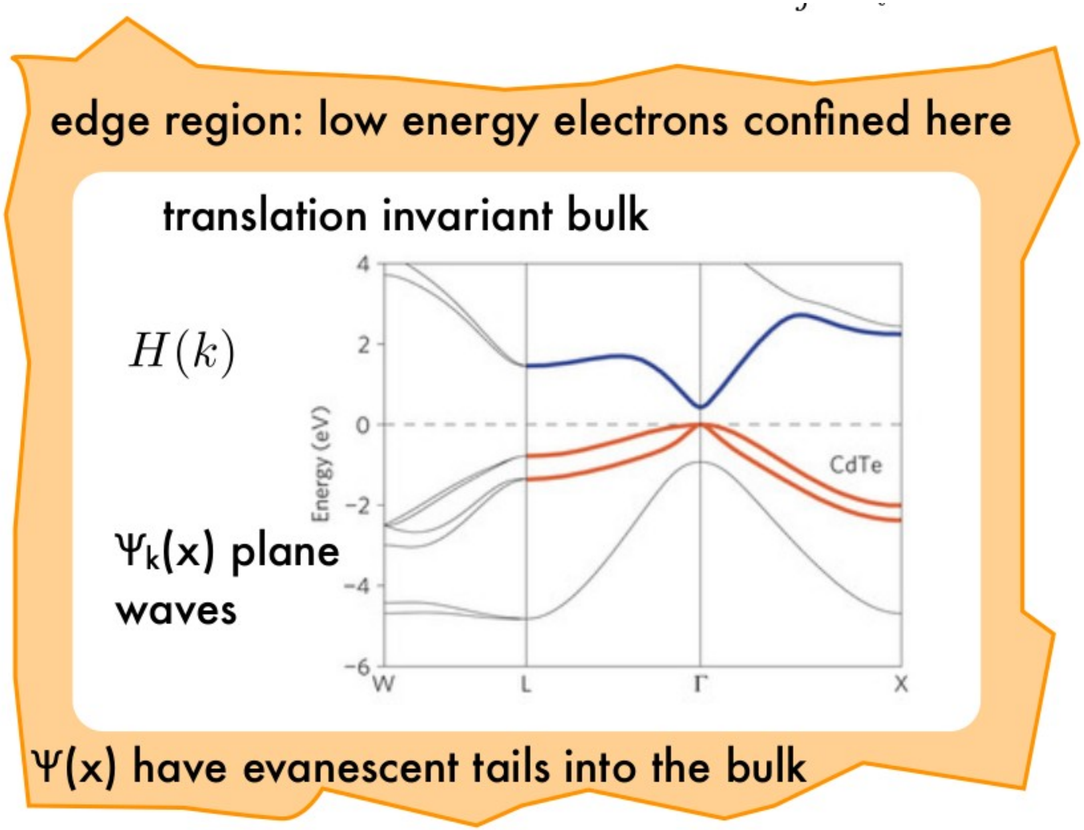
(includes superconductors in mean-field, using Bogoliubov-de Gennes trick)

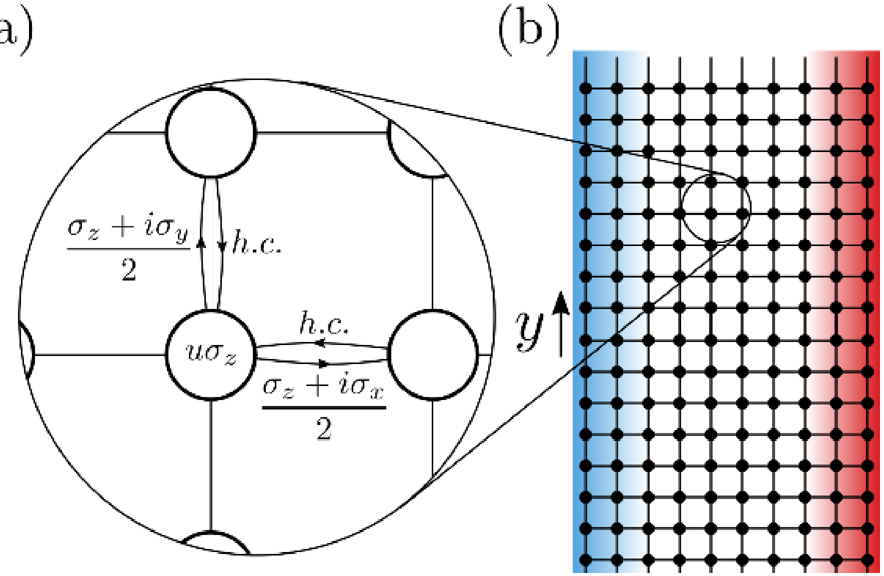


Bulk:
- -simple, can be clean,
- -most of the energy states
- -decides insulator/conductor

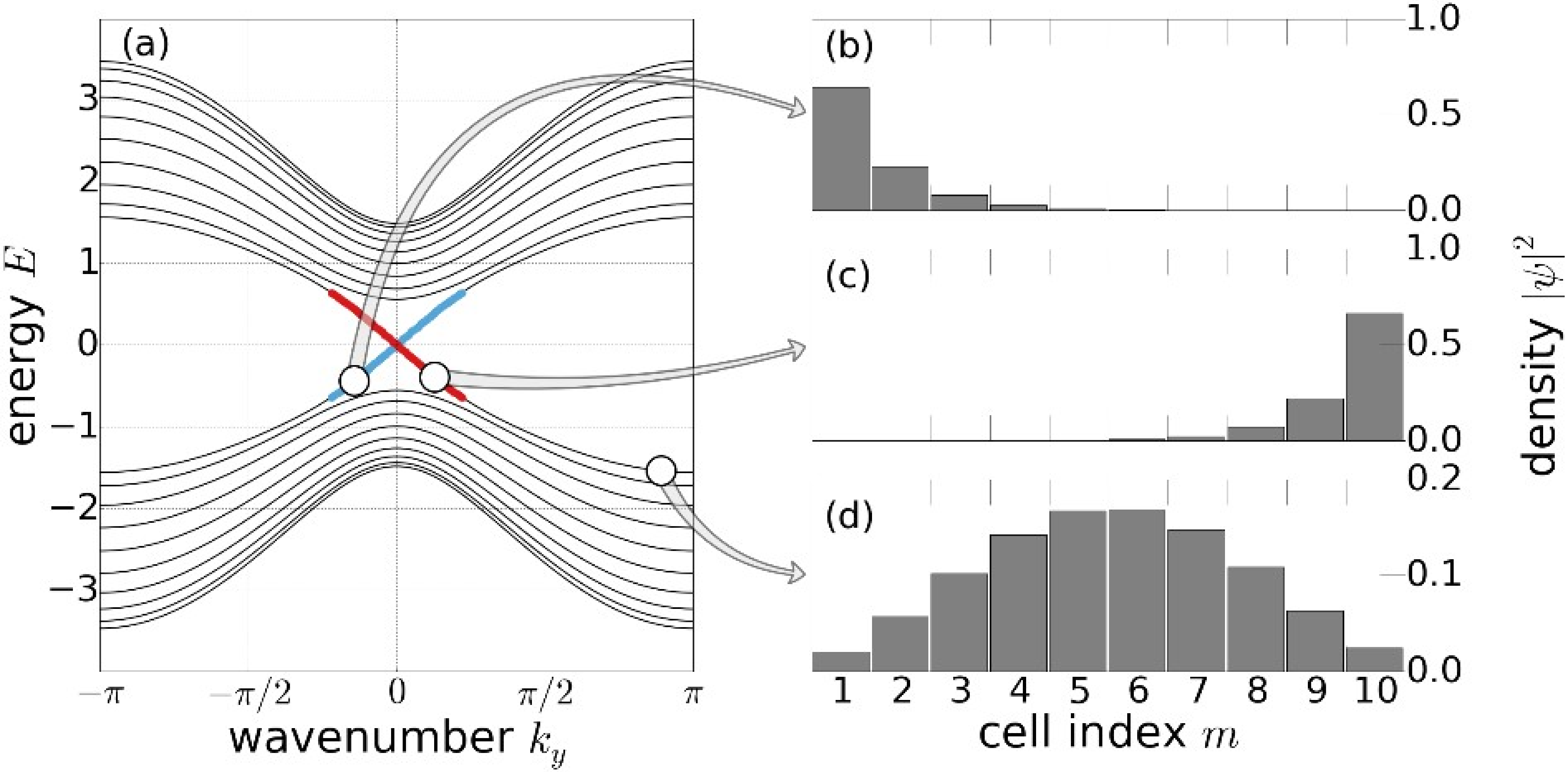
Boundary/edge:
- -disordered
- -few of the energy states
- -can hinder contact

Topological Insulator: has protected, extended midgap states on surface, which lead to robust, quantized physics





2D Chern Insulators: 1-way conducting states
→ no backscattering
→ perfect edge conduction



“Why call them *Topological* Insulators?”

a) Robust physics at the edge (2D: conductance via edge state channels) quantified by small integers

1D, quantum wire:
# of topologically protected
0-energy states at ends of wire

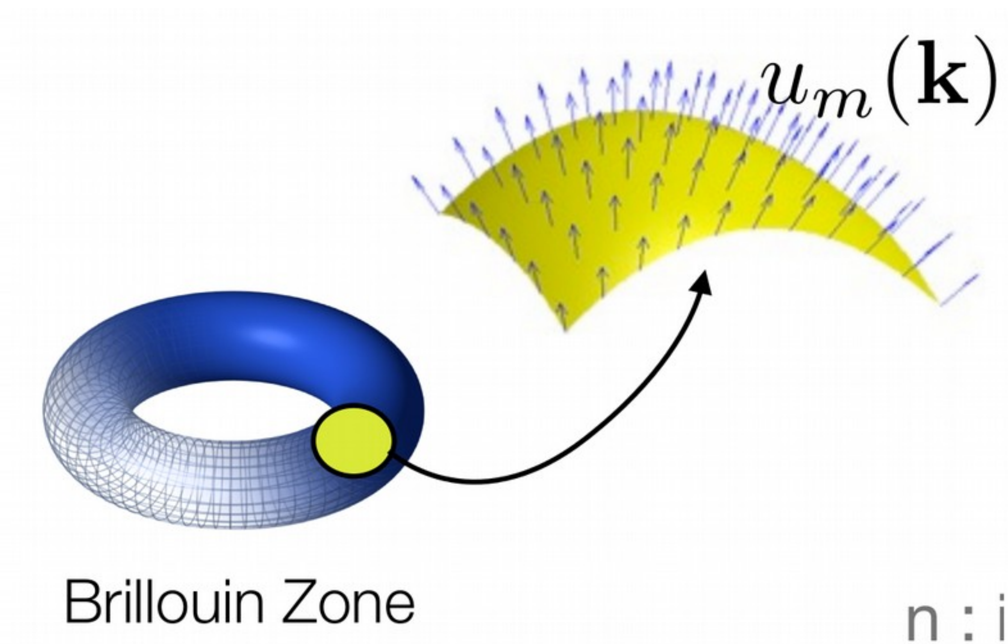
3D:
# of Dirac cones on surface

Cannot change by continuous deformation that leaves bulk insulating
→ TOPOLOGICAL INVARIANT

“Why call them *Topological* Insulators?”
b) Bulk description has a topological invariant, generalized “winding” in Brillouin Zone

Example: 2D, two levels: $\hat{H}(k) = \vec{h}(k)\hat{\vec{\sigma}}$

Mapping from d-dimensional torus to Bloch sphere



More general 2D: Chern number of occupied bands

$$A_\mu^{(n)}(k) = -i\langle n(k)|\partial_{k_\mu}|n(k)\rangle$$

$$F_{xy}^{(n)}(k) = \partial_{k_x}A_y^{(n)} - \partial_{k_y}A_x^{(n)}$$

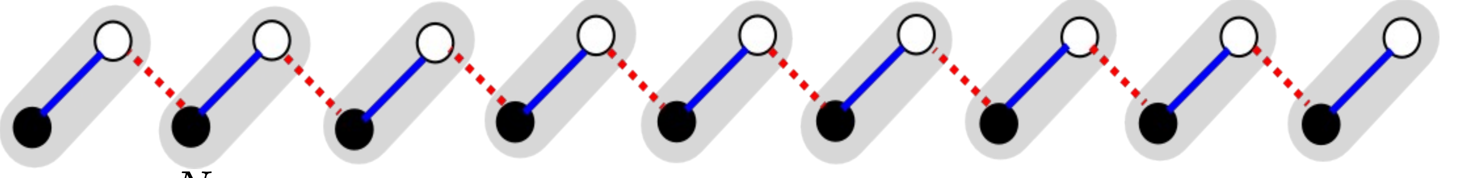
$$\boxed{Q^{(n)} = \frac{1}{2\pi}\int_{BZ} d^2k F_{xy}^{(n)}(k)}$$

Central, beautiful idea of Topological Insulators:
Bulk—boundary correspondence:
“winding number” of bulk = # of edge states

weeks 1-5:	gather tools, build intuition
week 6:	Central aim of the course: prove bulk—boundary correspondence for the 2-dimensional case
weeks 7-10:	generalize/understand

week 1: 1 Dimension, quantum wires with Sublattice Symmetry

Toy model (for polyacetylene):
Su-Schrieffer-**Heeger** (SSH, 1979), chemistry Nobel 2000



$$H = \sum_{j=1}^{N} \left(v_j |2j\rangle\langle 2j-1| + w_j |2j+1\rangle\langle 2j| \right) + h.c.$$

acquire familiarity with basic concepts:

- Edge States
- Topological invariant (Adiabatic deformations)
- Bulk Hamiltonian
- Bulk Invariant (winding number)
- Bulk—boundary correspondence through adiabaticity

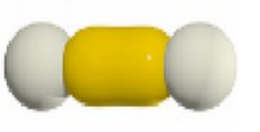
weeks 2,3: Gather mathematical tools:
Berry phase, Chern number, Polarization

Bulk polarization identified with Zak phase:

$$P = \frac{1}{2\pi i} \sum_{n<0} \int_{BZ} dk \langle n(k)| \frac{d}{dk} |n(k)\rangle$$

Projected to a single sublattice:

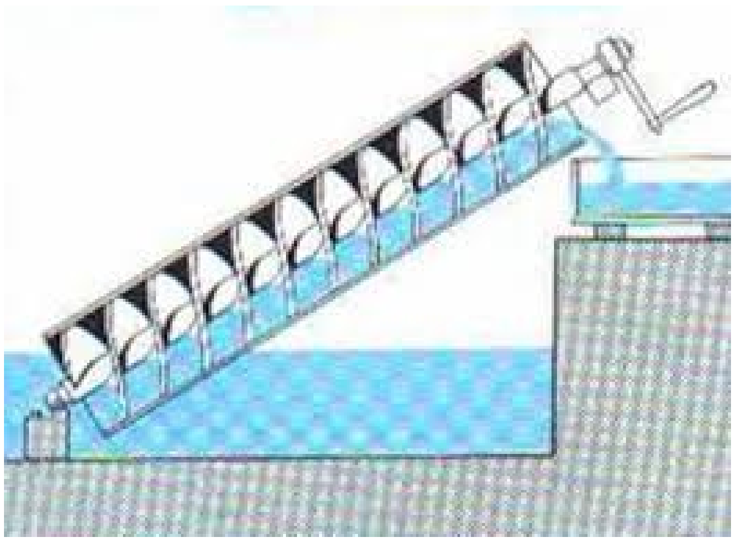
$$P_A = \frac{1}{2\pi i} \sum_{n<0} \int_{BZ} dk \langle n(k)| \Pi_A \frac{d}{dk} \Pi_A |n(k)\rangle$$



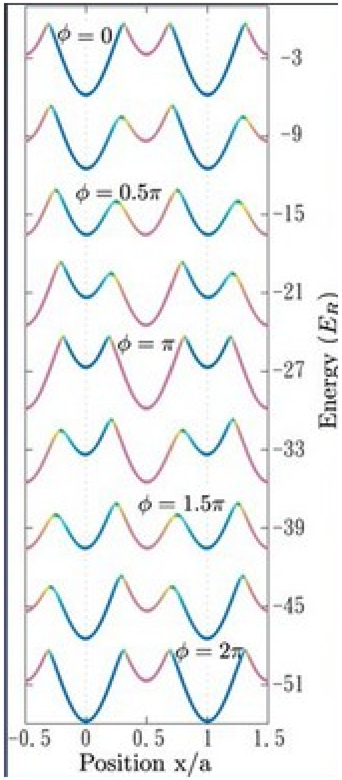
Sublattice polarization:

$$P_A - P_B = \frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log\det h(k) \equiv \nu[h]$$

weeks 4-5: Gather conceptual tool: Thouless Charge Pump

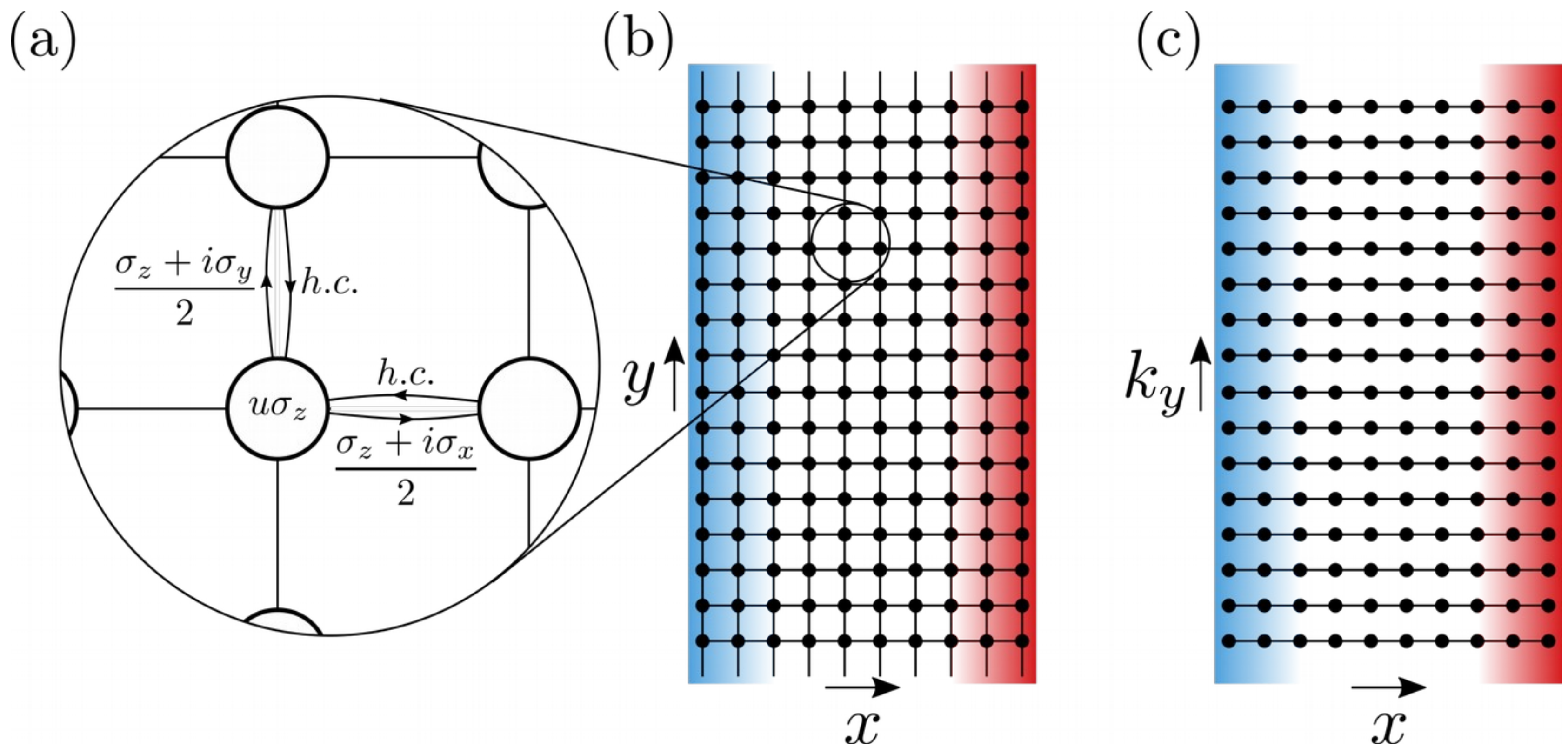


Archimedes screw:
displace water by
periodic pump
 - x liter per cycle



Thouless pump: displace charge
by periodic change in potential shape
 - n charges per cycle

week 6: Bulk—boundary correspondence for 2-dimensional Chern Insulators



Proof by mapping Chern Insulator to a Thouless pump
(a variant of dimensional reduction)

week 7: Continuum models of topological insulators

- Envelope Function Approximation
- No Brillouin Zone
- Simple analytical arguments

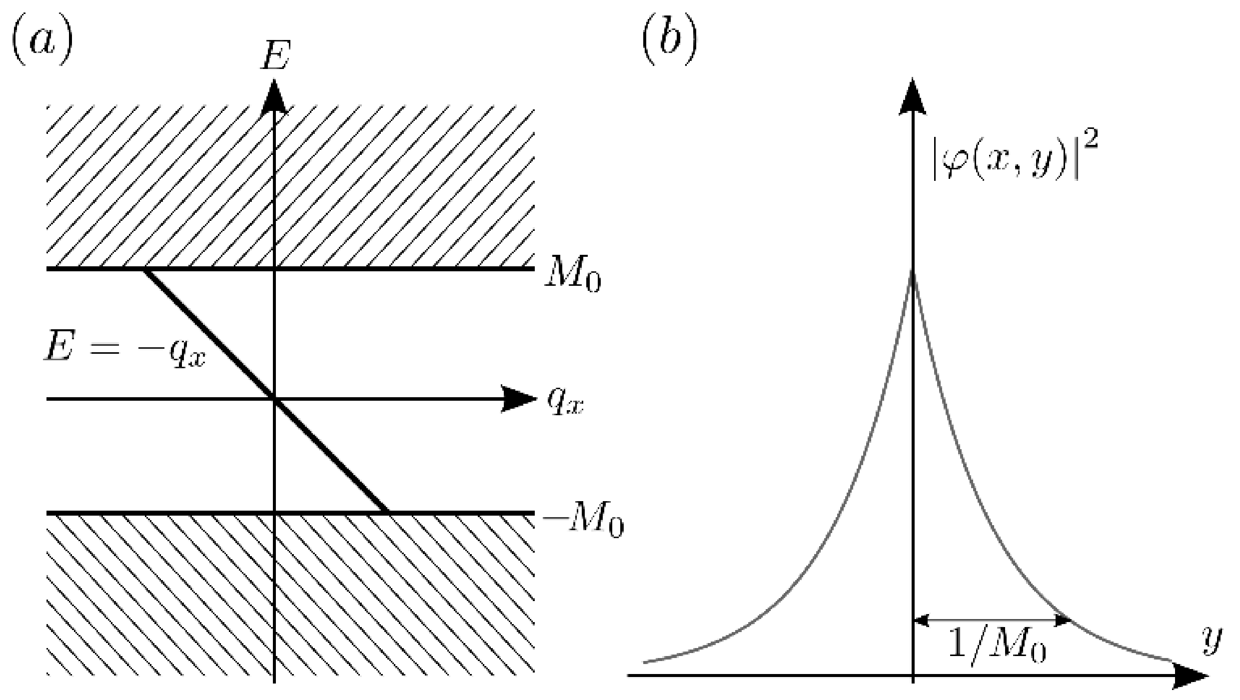
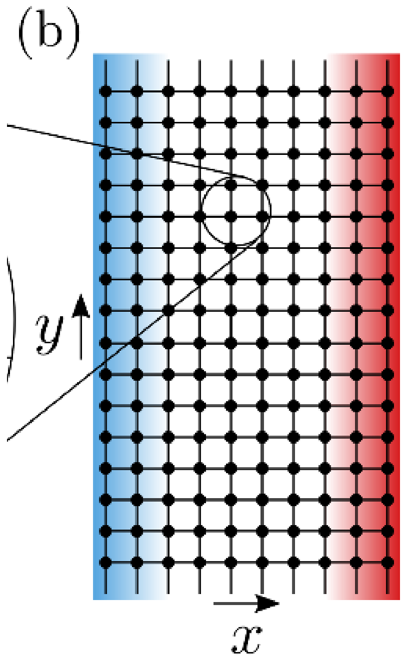
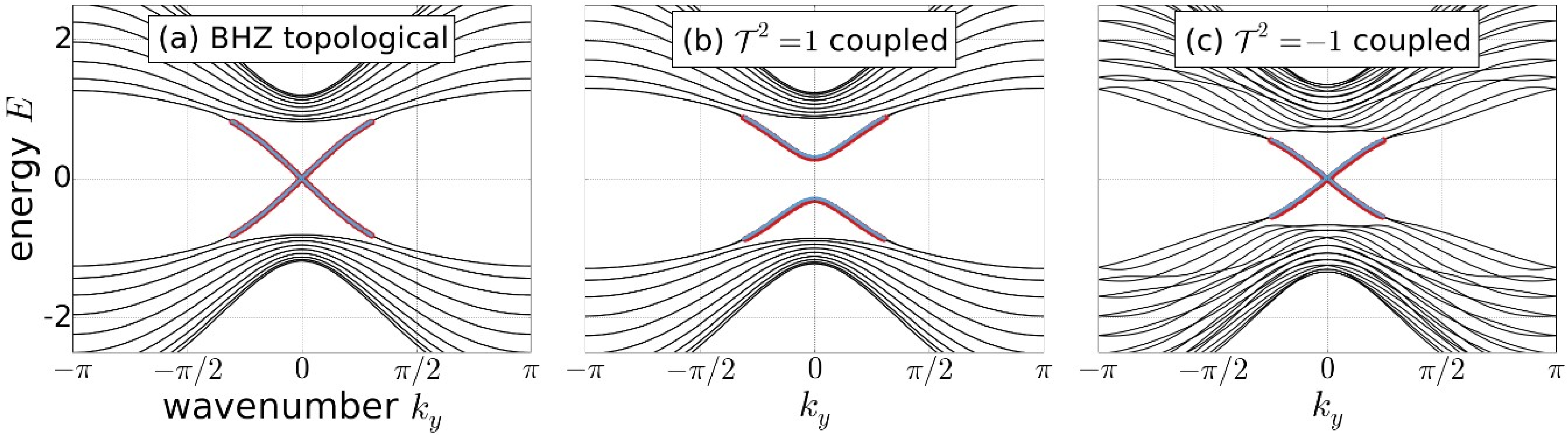


Figure 7.4: Chiral state obtained from the two-dimensional Dirac equation. (a) Dispersion relation and (b) squared wave function of a chiral state confined to, and propagating along, a mass domain wall.

weeks 8-9: Time-reversal-symmetric Topological Insulators

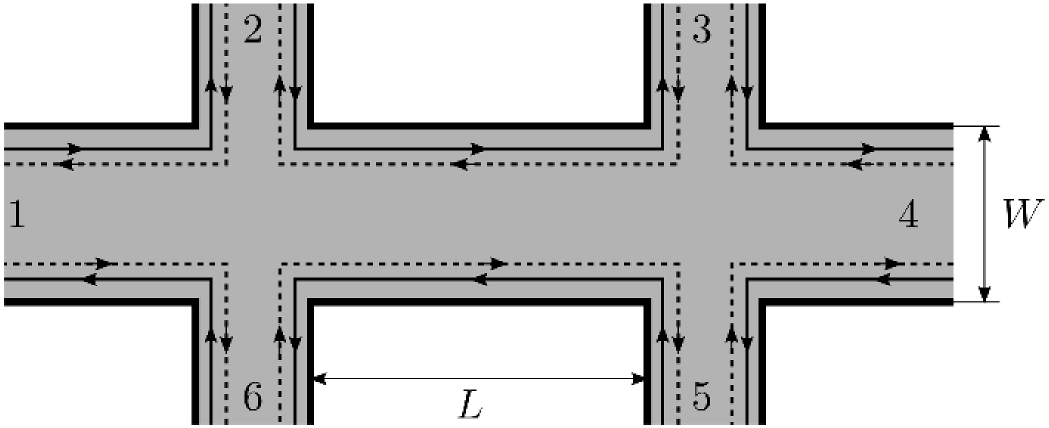
- Two types of time reversal
- Time reversal prevents one-way propagation (Chern=0)
- Kramers degeneracy
- Edge states protected by time reversal





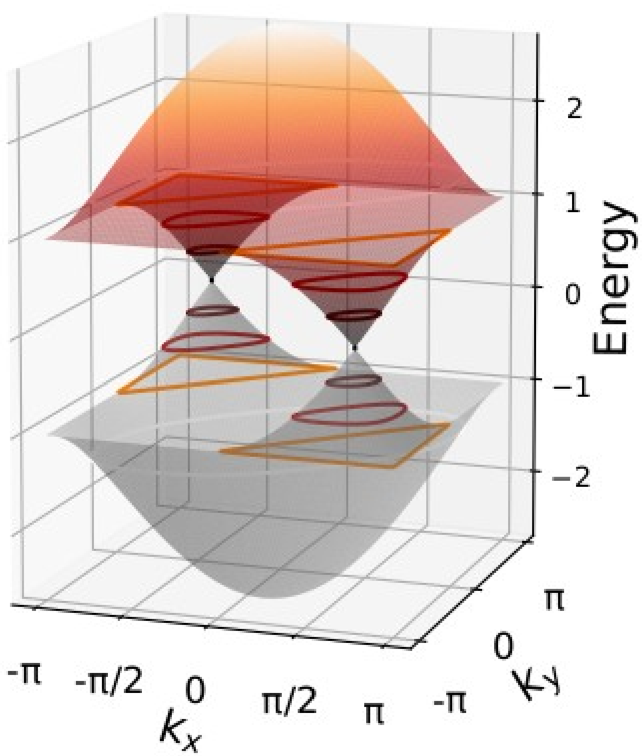
week 10: Electrical conduction as “smoking gun” signature of edge states: what it means, how it is measured

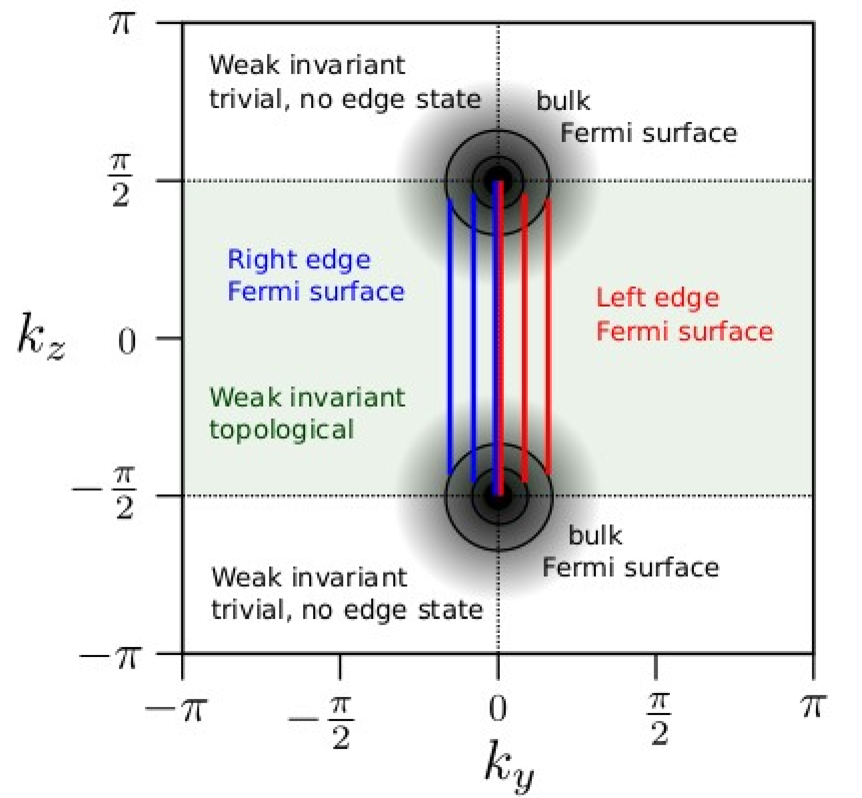
- Landauer--Büttiker picture of conductance
- Interpreting experiments
- Effects of decoherence



week 11 Topological semimetals, Weyl semimetals

- Topologically protected band crossings
- 2D: graphene, 3D: Weyl semimetals
- Surface Fermi arcs





weeks 12? If we have time at end of semester, explore extra material

Scattering theory of topological insulators, Green’s function formulation

More on experiments and model systems

Generalized topological invariants using differential geometry

Topologically protected states on topological defects

Next semester: Topological Superconductors

Bogoliubov—de Gennes

Majorana fermions in wires & 2D

Applications for quantum computing

Complete Periodic Table of Topological Insulators

Taste of Topological Order (interacting systems)

An example for how well developed the theory is: universality classes of Topological Insulators, “Periodic Table”

Symmetry Θ^2	Ξ^2	Π^2	$\delta = d - D$: 0	1	2	3	4	5	6	7
0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
−1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
−1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
−1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
0	−1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
1	−1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

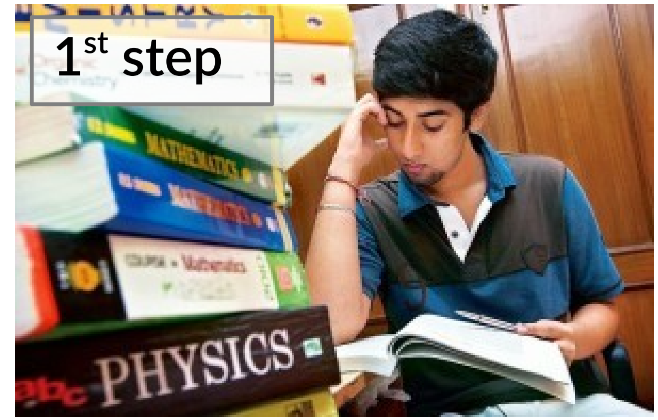
·Kitaev (2008)
·Schnyder, Ryu, Furusaki, Ludwig (2009)
·Teo & Kane, PRB 82, 115120 (2010)

Summary & motivation

- Band Insulators can have bulk topological invariants
- Universality: dimension, symmetries matter
- Bulk topological invariants predict edge states
- Systems of different dimensionality connected
- Useful for protection of quantum information
- Window into Topological Order

We teach this course using Peer Instruction

1. Prepare for class
   Read next section of lecture notes,
   (watch youtube lectures,
   discuss with friends, solve exercises)

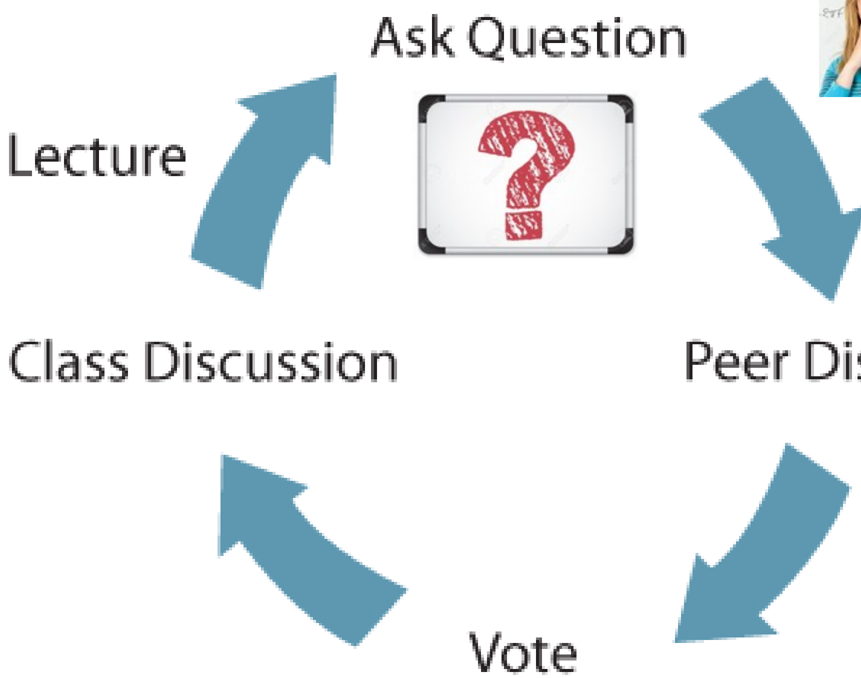


2. First part of class:
   we summarize,
   you ask your questions

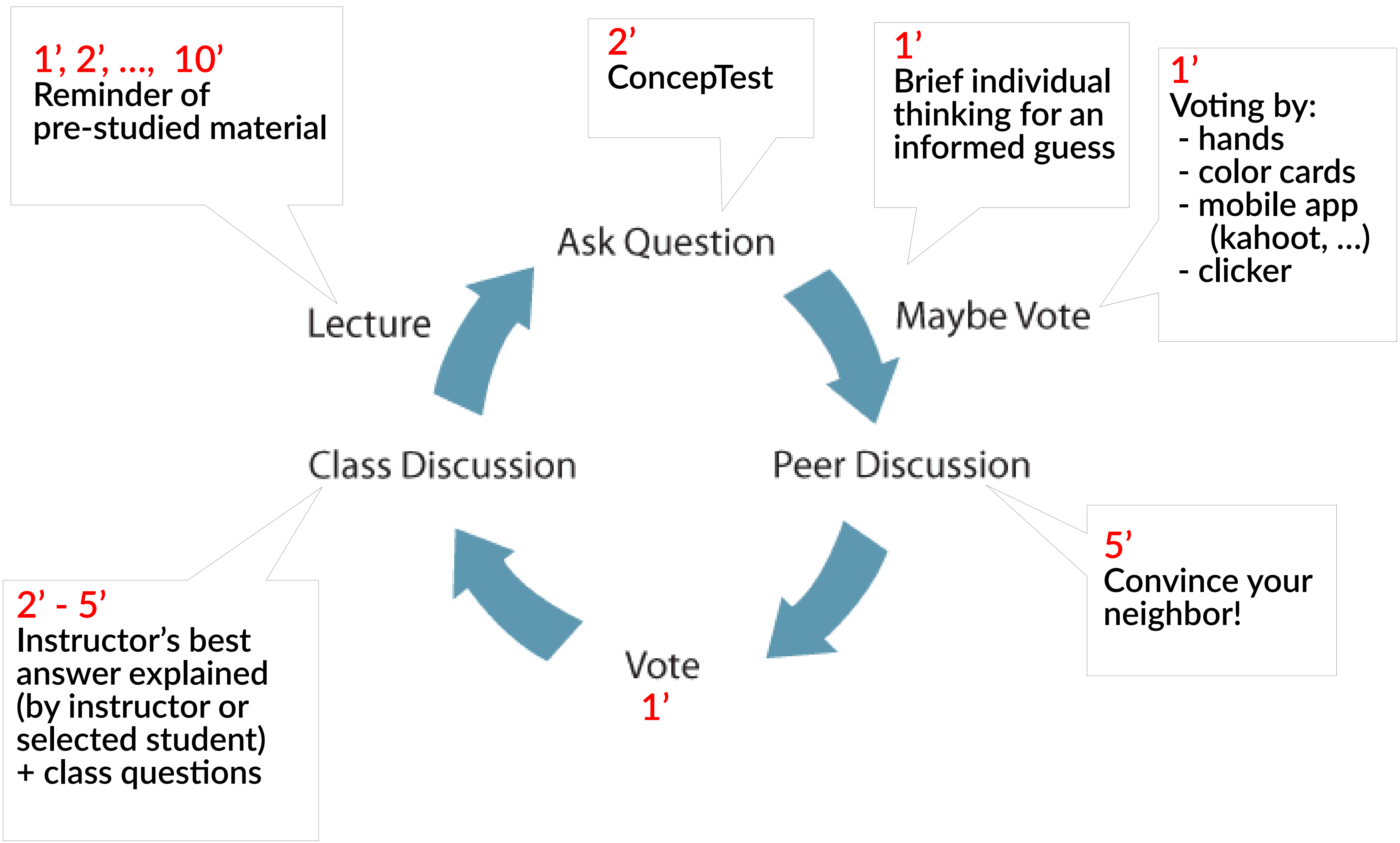
3. 2nd part of class: structured discussion

Peer Instruction: 10-15-min structured discussion, all students participate

Eric Mazur, Harvard professor (quantum optics)
- developed for premed Harvard course 1990
- improved continuously, large online community



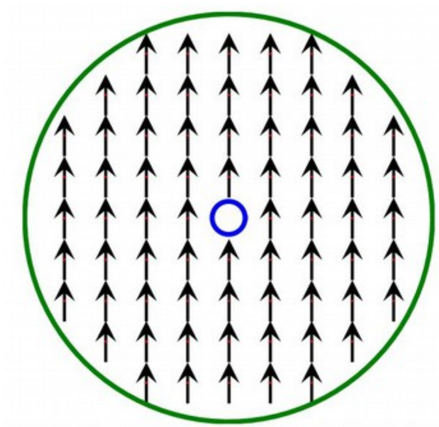
Peer Instruction: 10-15-min structured discussion, all students participate



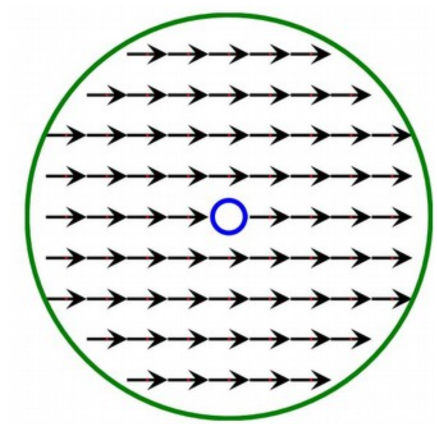
Example: 2-dimensional smooth vector fields on punctured disks (as in Kosterlitz-Thouless)

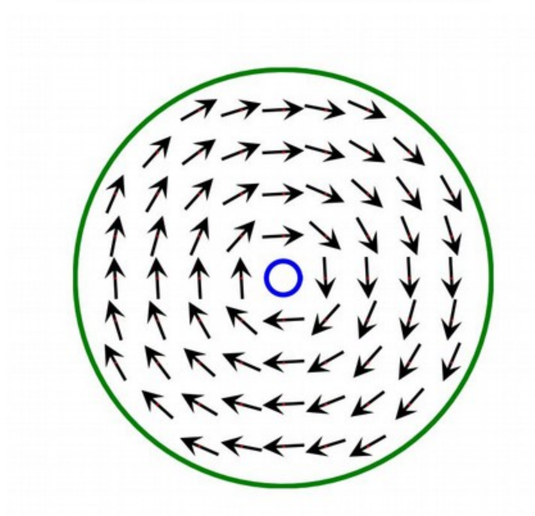
$$\mathbf{v}(\mathbf{r}) : \mathbb{R}^2 \to \mathbb{R}^2, \text{ but with } 0.1 < |\mathbf{r}| < 1 \text{ and } \forall \mathbf{r} : |\mathbf{v}(\mathbf{r})| = 1$$

$\simeq$ homotopic equivalence: v(r)$\simeq$w(r) iff they can be connected continuously

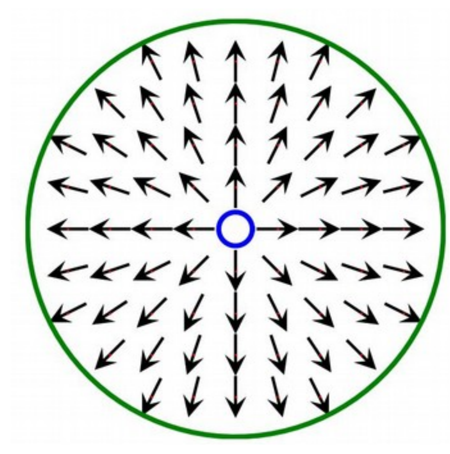


$\simeq$





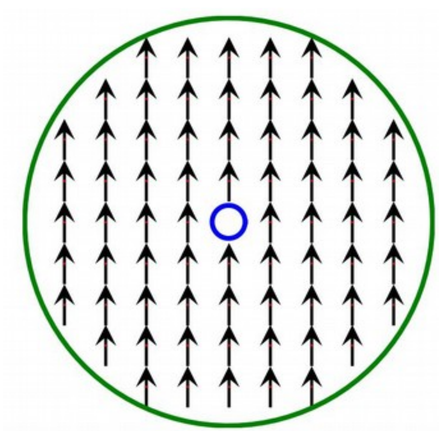
$\simeq$



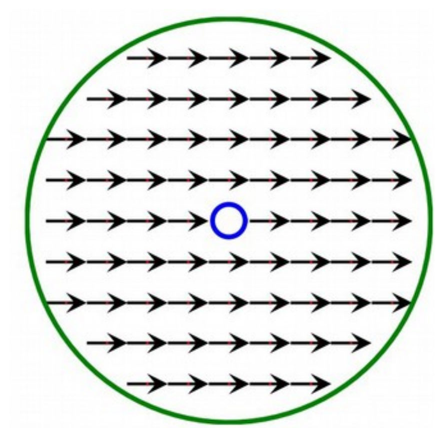
Example: 2-dimensional smooth vector fields on punctured disks (as in Kosterlitz-Thouless)

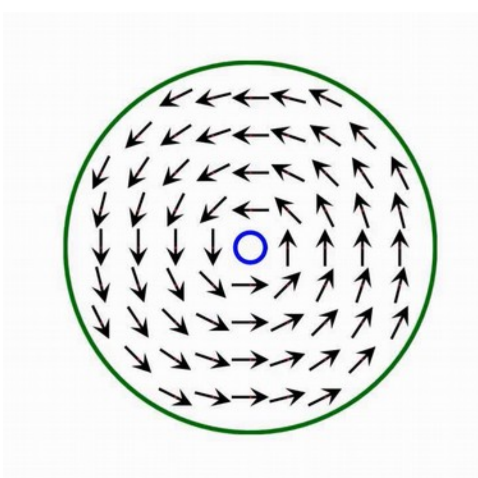
$$\mathbf{v}(\mathbf{r}) : \mathbb{R}^2 \to \mathbb{R}^2, \text{ but with } 0.1 < |\mathbf{r}| < 1 \text{ and } \forall \mathbf{r} : |\mathbf{v}(\mathbf{r})| = 1$$

$\simeq$ homotopic equivalence: v(r)$\simeq$w(r) iff they can be connected continuously

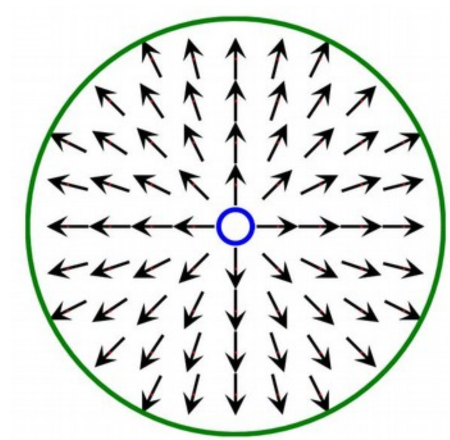


$\simeq$



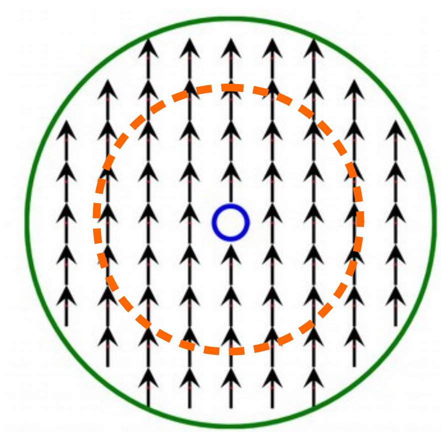


$\simeq$

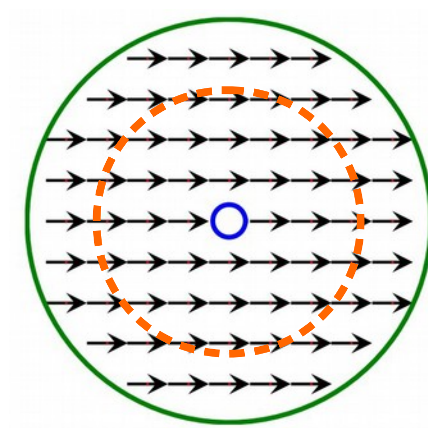


2-dimensional smooth vector fields and winding numbers

Winding number N of v(r) along a closed loop is topological invariant: obstruction for continuous deformations

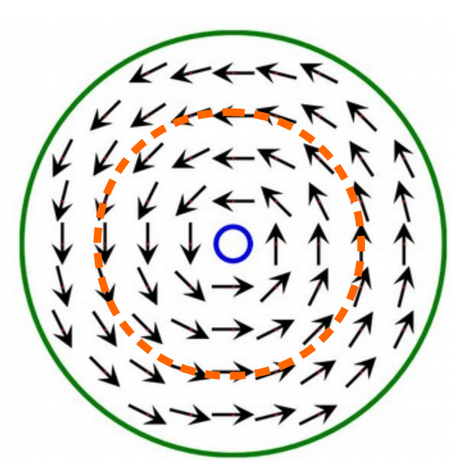


$\simeq$

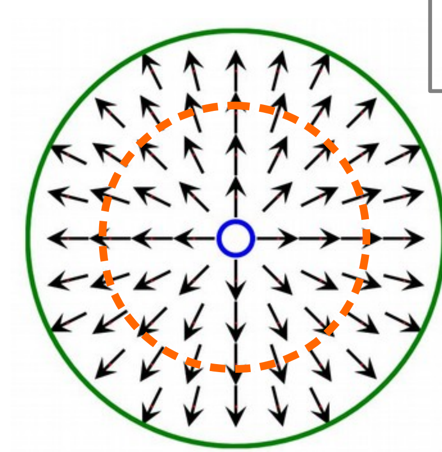


N = 0

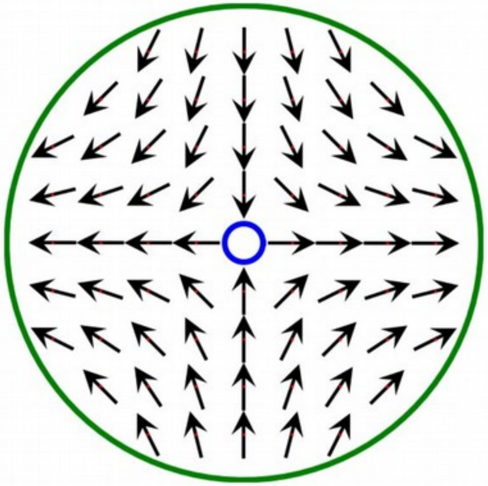
$$N = \frac{1}{2\pi} \oint \mathbf{v}(l) \times \frac{\mathrm{d}\mathbf{v}(l)}{\mathrm{d}l} \mathrm{d}l$$

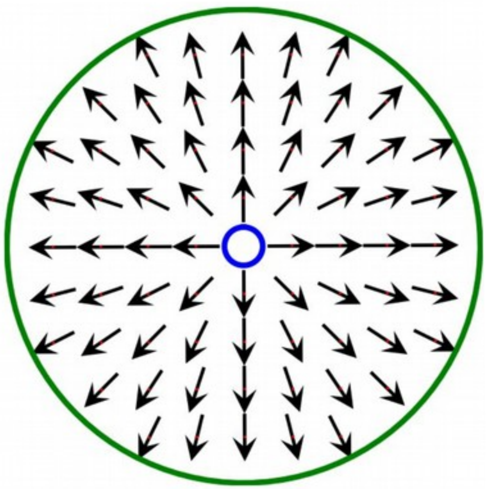


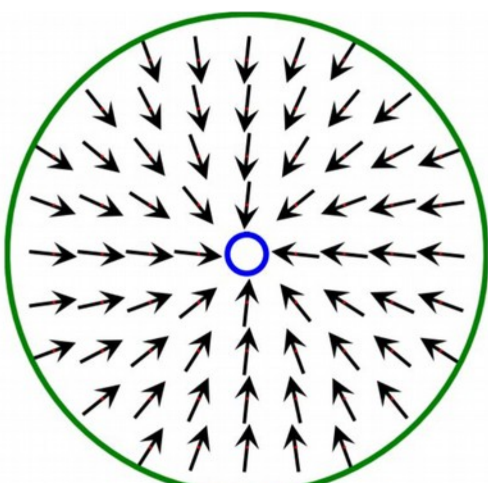
$\simeq$



N = 1







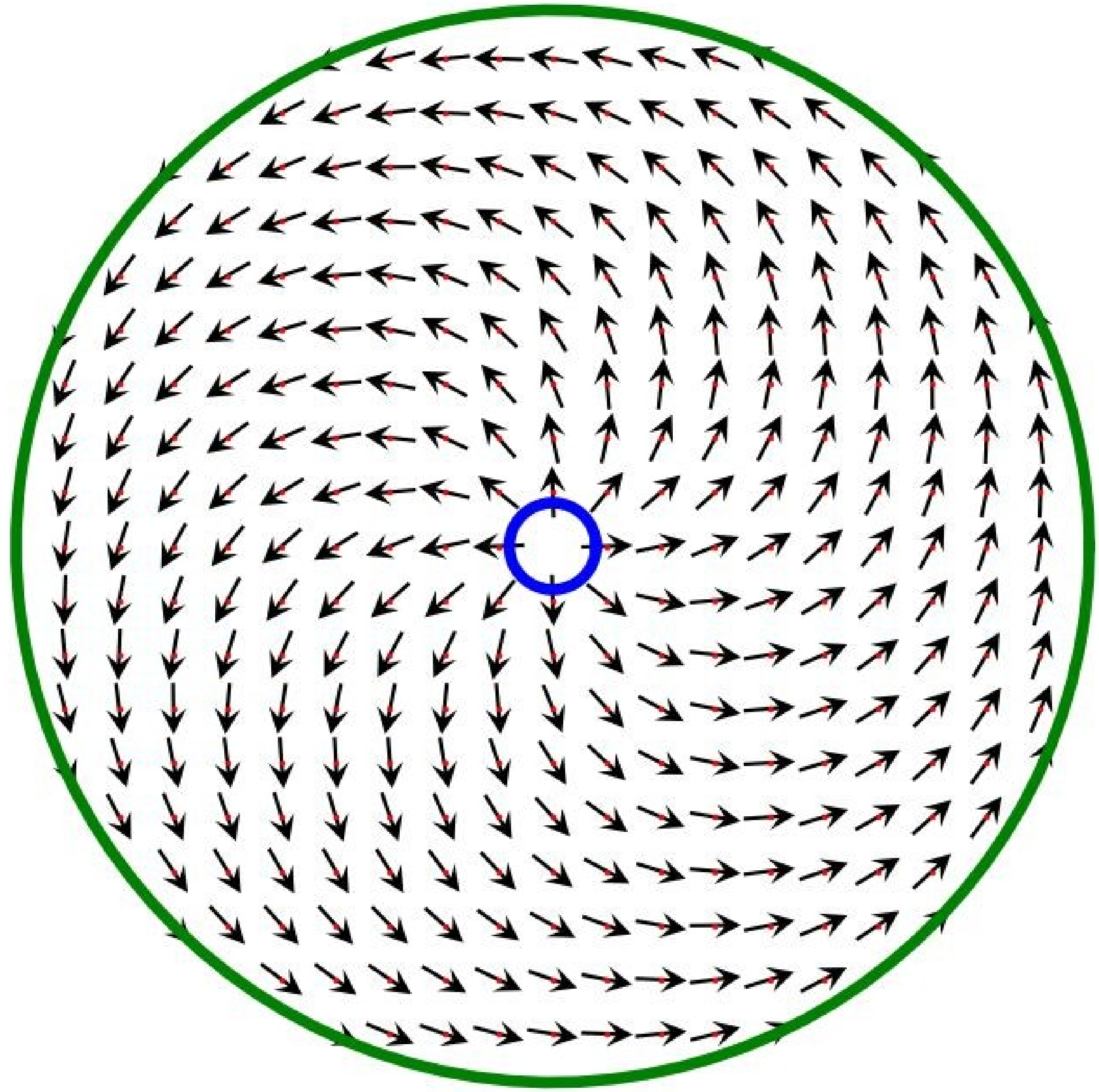
How many of the small ones is ≃ to the big one?

A) 0?

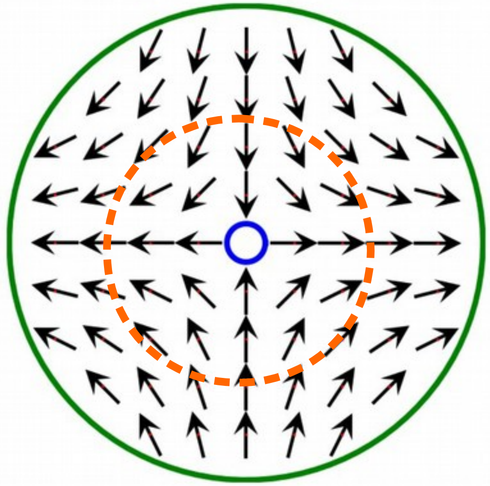
B) 1?

C) 2?

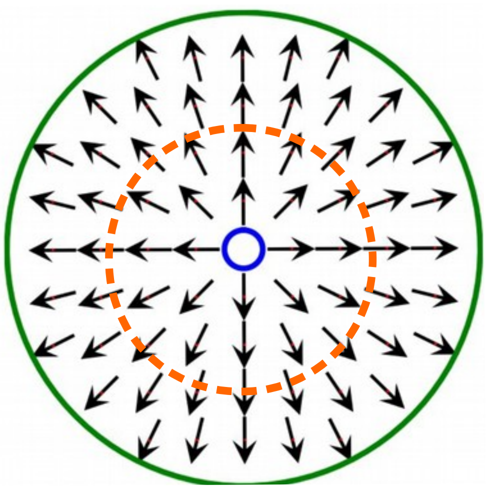
D) 3?



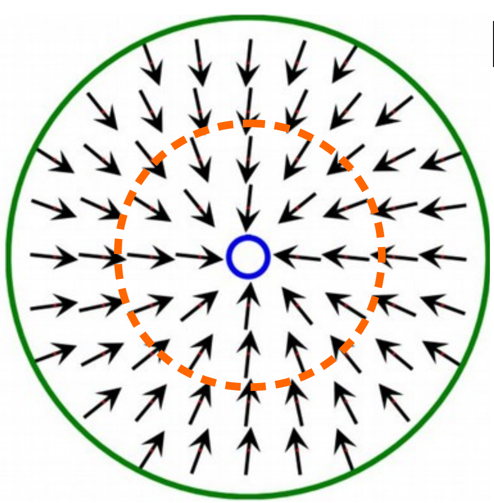
Answer: 2 are ≃ to the big field
(1. calculate winding number)



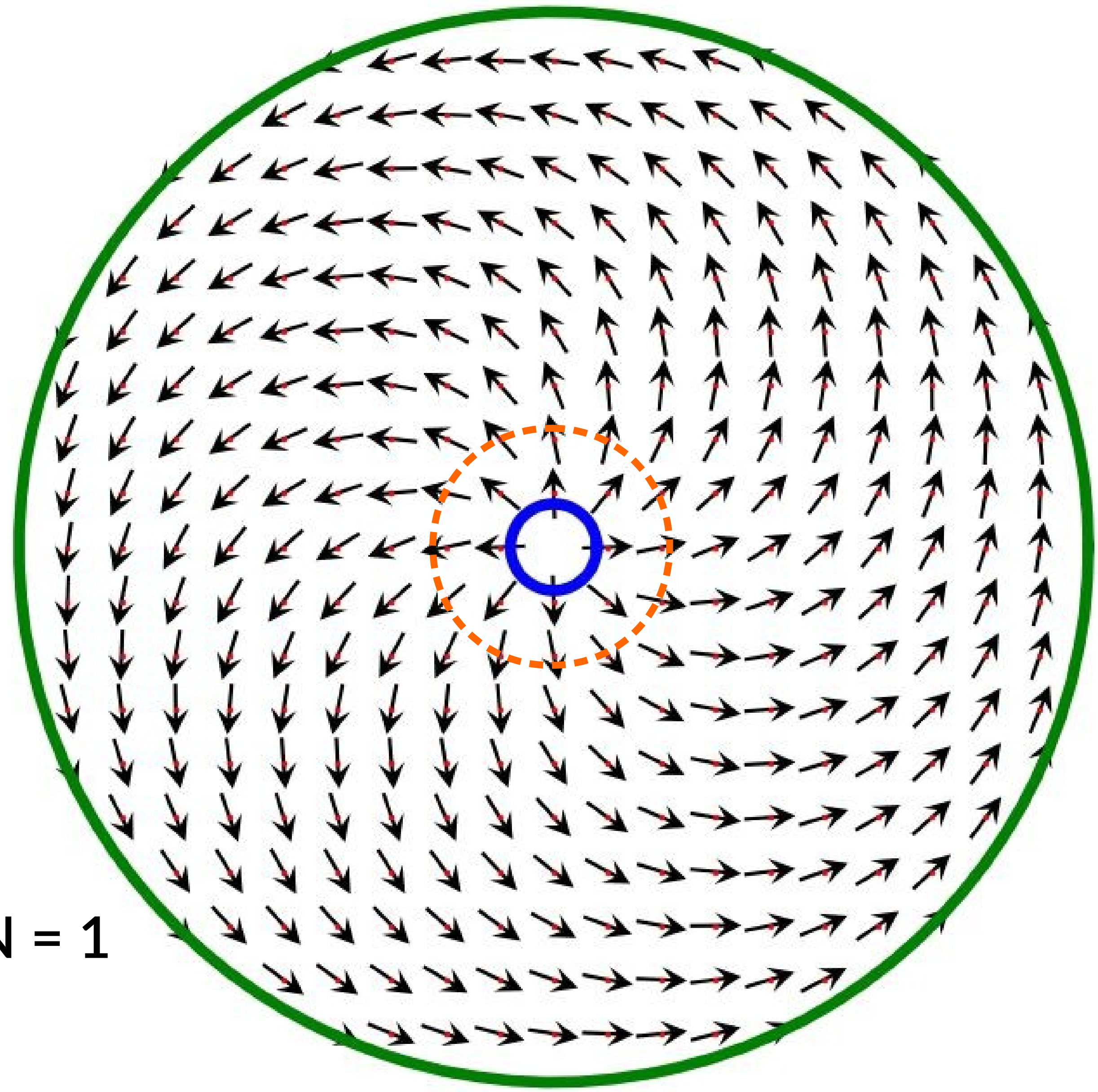
N = -1



N = 1

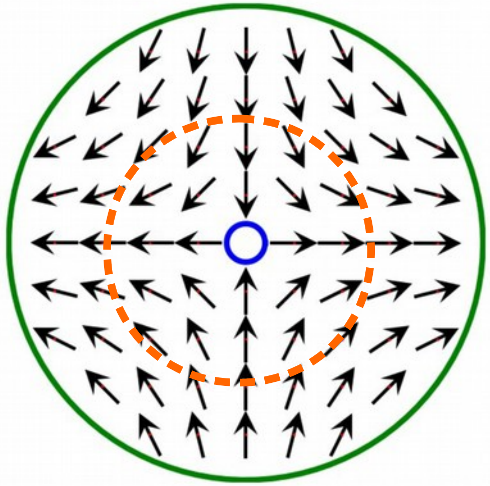


N = 1

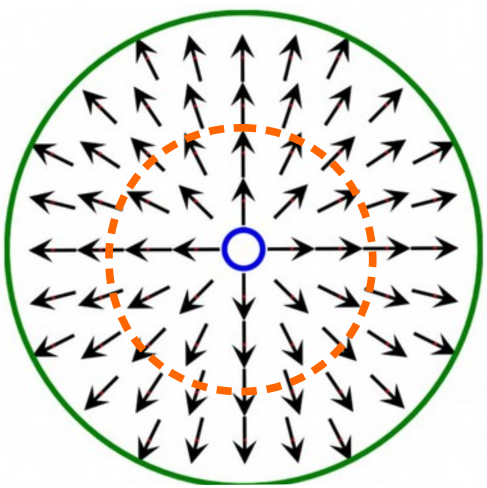


N = 1

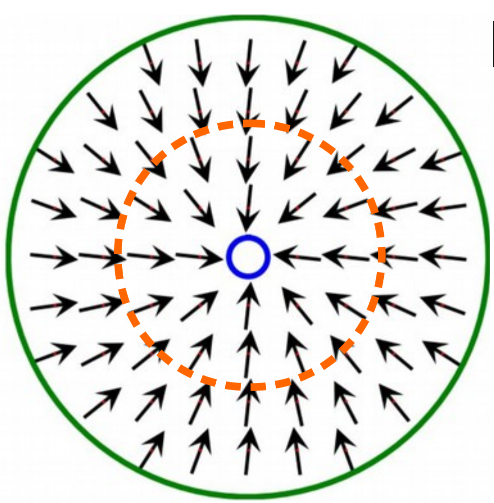
Answer: 2 are ≃ to the big field
(2. show animation)



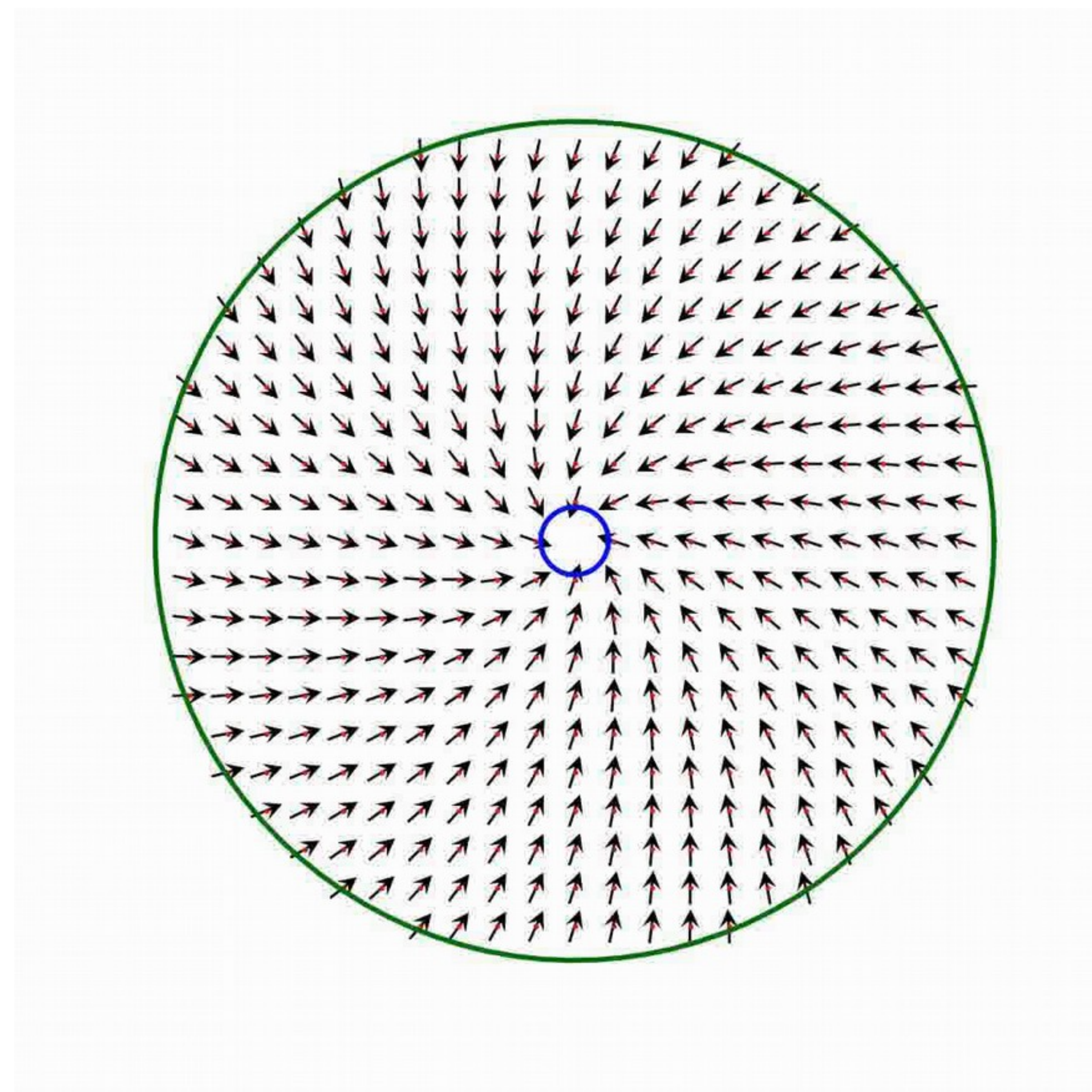
N = -1



N = 1



N = 1



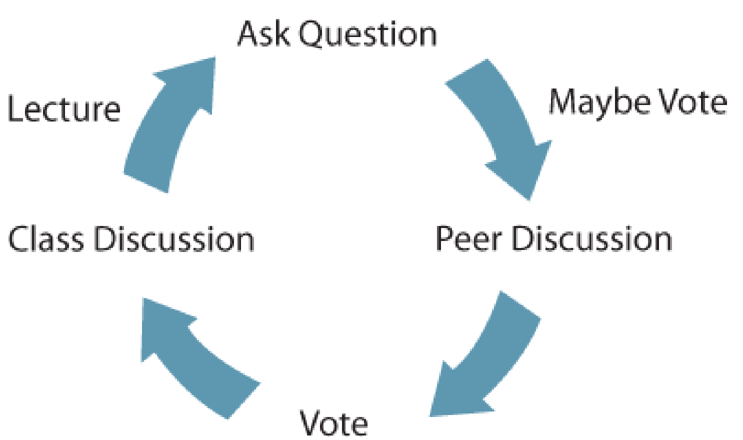
Peer Instruction makes lecturing (more) useful

Fun game for students

- Breaks monotonicity
- Engages high-achieving and underachieving students
- Develops communication skills, self-confidence
- Gives real-life understanding
- Pre-lecture reading needed

Useful feedback for instructor

- Allows to shape course
- Voting: Instant feedback about whole group
- Listening in to discussions: individual problems
- ConcepTests needed (many online)



Peer Instruction Model by Eric Mazur

If you put some energy into this Topological Insulators course during semester, this will be fun!

you need to:

- *Read ahead in the lecture notes (on website)*
- *Participate in classroom*
- *Feel free to experiment with python scripts (on website)*

http://physics.bme.hu/BMETE11MF34_kov

you obtain:

+ Develop deep understanding of topic before the exam period
+ Develop communication skills