On Black Holes and Entropy

an introductory talk from an amateur admirer

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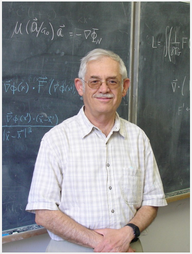
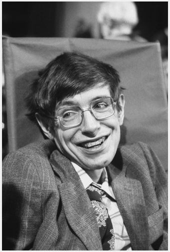
2019. 10. 04.



$$S_{\mathrm{BH}} = \frac{Ac^3}{4\hbar G}$$



Credits

Bekenstein, Jacob D., ”Black holes and entropy”, (1973), 10.1103/PhysRevD.7.2333

Black holes

1. A **black hole** is a **region of spacetime** that's gravitational acceleration is so strong $\rightarrow$ no particle or EM wave can escape it.
2. No-hair theorem: they can be described **only** by three externally observable parameters:
   1. mass M
   2. charge Q
   3. angular momentum L

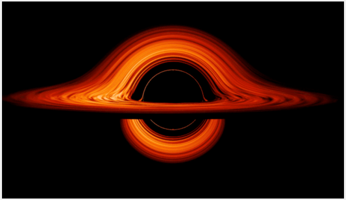


Figure: NASA's visualization of a black hole.

Black hole types

Black holes are categorized by their external properties:

	Non-rotating ($L = 0$)	Rotating ($L \neq 0$)
Uncharged ($Q = 0$)	Schwarzschild	Kerr
Charged ($Q \neq 0$)	Reissner–Nordström	Kerr–Newman

If two black holes share the same (M, L, Q) values, they are said to be non differentiable. (No-hair theorem.)

Internal structure: How the black hole was formed. (e.g. from a white dwarf, neutron star, etc.)

Two black holes with different internal structures but same (M, L, Q) parameters are the same!

The total mass

The total mass-energy of the black hole is given by:

$$M^2 = \frac{L^2}{4M_{\text{ir}}^2} + \left(\frac{Q^2}{4M_{\text{ir}}} + M_{\text{ir}}\right)^2$$

where

1. L is the rotational energy
2. Q is the Coulomb energy
3. M_{ir} is irreducible mass-energy: this is the energy that **can not be extracted** (through 'Penrose processes').

Black holes and processes

1. Black holes increase their horizon undergoing any processes. (Floyd and Penrose, conjecture)
2. A capture of a particle by a Kerr black hole can never end up lowering the **irreducible mass**. (Christodoulou)

$$M_{\text{ir}} = \sqrt{A/16\pi}$$

3. Reversible processes $\rightarrow$ M_{ir} doesn't change.
4. A black hole's surface can not decrease in *any* process. (Hawking, general theorem, holds for a system of black holes as well)

area $\Leftrightarrow$ entropy analogy? $\Rightarrow$ black hole statistical physics?

Analogies

i Entropy always increasing. $\Leftrightarrow$ Black hole area always increasing.

ii Degradation of energy: entropy increasing = less energy can be converted to work. $\Leftrightarrow$ The irreducible energy can't be extracted.

$$M \geq M_{\text{ir}} = \sqrt{A/16\pi}$$

iii Separate thermodynamic systems in equilibrium can do work when interacting. $\Leftrightarrow$ For a system of Schwarzschild black holes ($M = M_{\text{ir}}$) merging:

$$E_d = \sqrt{\sum A/16\pi} = \sqrt{\sum M_{\text{ir}}^2}$$

A formal definition of black hole entropy

- Black hole analog of the first law: $dE = TdS - pdV$.
- Rationalized area: $\alpha = A/(4\pi)$
- Kerr black hole with (M, Q, L) - rationalized area:

$$\alpha = r_+^2 + a^2 = 2Mr_+ - Q^2$$

where $a = L/M$; $\quad r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$

- differentiation and manipulation:

$$dM = \underbrace{\Theta d\alpha}_{\text{entropy term}} + \underbrace{\Omega dL + \Phi dQ}_{\text{work term}}$$

$$\Theta = \tfrac{1}{4}(r_+ - r_-)/\alpha$$
$$\Omega = a/\alpha$$
$$\Phi = Qr_+/\alpha$$

$\rightarrow \alpha$ = Black hole entropy

Information and entropy

1. Entropy: lack of information about the actual internal configuration of the system. Shannon's formula for the entropy:

$$S = -\sum_n p_n \log p_n$$

2. Information = constraints on (p_n) $\rightarrow$ entropy is decreased

$$\Delta I = -\Delta S$$

   e.g. isothermic compression of ideal gas

3. Unit of information: bit (when a yes/no question is answered)

4. Second law of thermodynamics: entropy is increasing in a system towards equilibrium = washing out of the effects of initial conditions.

Black hole ”thermodynamics”

For information: black hole = thermodynamic system

Black hole entropy

Black hole entropy measures the **inaccessibility** of internal configurations which realize the ”thermodynamic” (externally observable) variables in equilibrium:

$$(p, T, V) \leftrightarrow (M, L, Q)$$

Internal configuration = how the black hole was formed

Black hole entropy $\neq$ thermodynamic entropy!

Construction of S_{BH}

Programme

i. We assume: $S_{\text{BH}} = f(\alpha)$ where f is monotonically increasing.

ii. We want S_{BH} to be valid for any, even dynamically evolving black hole. Expection: loss of information about initial conditions $\rightarrow$ gradual increase in S_{BH}.

iii. Hawking's theorem supports the choice of f.

e.g. a possible choice:

$$f(\alpha) \propto \sqrt{\alpha}$$

Claim: merging of two Schwarzschild black holes prohibits this.

$(S_{\text{BH}} \propto M_{\text{ir}} = M_{\text{Sch}})$

Expression for the black hole entropy

A simple choice: $f(\alpha) = \gamma\alpha$. $\rightarrow$ no contradiction!

Dimensional analysis: $\gamma = \eta\hbar^{-1}$; η dimensionless. Quantum nature!

How to get η?

1. We can't cause of quantum nature. BUT.
2. Throw particle into a Kerr black hole $\rightarrow$ information lost.
3. What's the minimum surface increase? That's exactly 1 bit of entropy increase. $\rightarrow$ integration gives back f.

Expression for the black hole entropy

Lengthy calculations $\rightarrow$ for a spherical particle with radius b and rest mass μ:

$$(\Delta\alpha)_{\text{min}} = 2\mu b$$

(independent of black hole parameters (M, L, Q))

What is b?

b = Compton wavelength ($\mu\hbar^{-1}$)

$\Rightarrow$ no internal structure! $\rightarrow$ 1 bit information ("exists or not?")

$\Rightarrow$ smallest entropy increase

quantum effects set a bound of

$$(\Delta\alpha)_{\text{min}} = 2\mu\hbar\mu^{-1} = 2\hbar$$

Expression for the black hole entropy

Thus:

$$(\Delta S_{\mathsf{BH}})_{\mathsf{min}} = (\Delta\alpha)_{\mathsf{min}}\frac{df}{d\alpha} = \log 2 \quad \Rightarrow \quad f(\alpha) = \left(\frac{1}{2}\log 2\right)\hbar^{-1}\alpha$$

Giving us

The black hole entropy

$$S_{\mathsf{BH}} = \left(\frac{1}{2}\log 2\right)\hbar^{-1}\alpha$$

in conventional units:

$$S_{\mathsf{BH}} = \left(\frac{\log 2}{8\pi}\right)\frac{k_B c^3 A}{\hbar G}$$

Black hole "temperature": $T_{\mathsf{bh}}^{-1} = (\partial S_{\mathsf{BH}}/\partial M)_{\mathsf{L,Q}} = (2\hbar/\log 2)\Theta$

Generalized second law of thermodynamics

Thought experiment:
Body containing common entropy goes down the black hole $\rightarrow$ we can't tell whether the total common entropy decreased in the process.

However, the **black hole entropy** compensates.

Generalized second law

The common entropy in the black hole exterior plus the **black hole entropy** never decreases.

$$\Delta S_{\mathrm{BH}} + \Delta S_{\mathrm{c}} = \Delta(S_{\mathrm{BH}} + S_{\mathrm{c}}) > 0$$

Note: we neglected statistical fluctuations.

Black hole thermodynamics

First law

$$dM = \Theta d\alpha + \Omega dL + \Phi dQ$$

The black hole entropy

$$S_{\mathsf{BH}} = \left(\frac{\log 2}{8\pi}\right) \frac{k_B c^3 A}{\hbar G}$$

Generalized second law

$$\Delta S_{\mathsf{BH}} + \Delta S_{\mathsf{c}} = \Delta(S_{\mathsf{BH}} + S_{\mathsf{c}}) > 0$$